

Approximation of Fixed Points In CAT(0) Spaces Using A New Three-Step Iteration Process For Asymptotically Quasi-Nonexpansive Mapping

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Abstract:

In this paper, we introduce a new three-step iteration scheme and establish convergence results for the approximation of fixed points of asymptotically quasi-nonexpansive mapping in the framework of CAT(0) space and also give an example of a nonlinear function that is asymptotically quasi-nonexpansive but it is not quasi-nonexpansive. Finally, we display the efficiency of the proposed scheme compared to different iterative schemes by a numerical example in the literature. Our results generalize and improve many well-known results in the literature of iterations in fixed point theory.

Keywords: CAT(0) space; asymptotically quasi-nonexpansive mapping; fixed point; iteration process.

1. INTRODUCTION

In order to approximate the solution of a class of problems, iteration methods, as we know, are numerical algorithms that compute a sequence of more accurate iterates. These techniques are practical applications of applied mathematics that can be used to address issues in a variety of fields, including biology, transportation, network analysis, economics, and finance.

A thorough examination of the qualitative characteristics of iteration algorithms, including convergence, stability, error propagation, and stopping criteria, is necessary when designing them. Research in this field is ongoing, with several eminent scientists worldwide have contributed to and continuing to focus on the qualitative analysis of iteration techniques; please, see Ishikawa [24], Mann [30], Noor et al. [16, 17, 18], Ćirić et al. [12, 13, 14], Kirk and Shahzad [29], Ofoedu et al. [5, 4], Shahzad and Zegeye [20], Yao et al. [31, 32]. Special emphasis is given to studies on CAT(0) spaces: like, Abbas et al. [19], Saluja [8, 6, 7], Shahzad [21], Chang et al. [26], Dhompongsa and Panyanak [27].

Assume that (X, d) is a metric space. A geodesic path between $\theta \in X$ and $\eta \in X$ (or, to put it another way, a geodesic from θ to η) is a map r from $[0, l]$ to X with $r(0) = \theta$, $r(l) = \eta$ and $(d(r(t), r(t_0))) = |t - t_0|$ for any $t, t_0 \in [0, l]$. Thus r is an isometry and $d(\theta, \eta) = l$. The image of r is a geodesic (or metric) segment that joins θ and η . When geodesic is unique, it is denoted by $[\theta, \eta]$.

The space (X, d) is said to be a geodesic space if every two points in X are connected by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic between θ and η for any θ, η in X . If D contains every geodesic segment connecting any two points, the subset $D \subseteq X$ is convex.

A geodesic triangle $\Delta(\theta_1, \theta_2, \theta_3)$ in a geodesic metric space (X, d) is made up of three points $\theta_1, \theta_2, \theta_3$ in X (the vertices of Δ), with a geodesic segment connecting each pair of vertices (the edge of Δ).

A comparison triangle for the geodesic triangle $\Delta(\theta_1, \theta_2, \theta_3)$ in (X, d) is a triangle $\bar{\Delta}(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)$ in Euclidean space R^2 such that $d_{R^2}(\bar{\theta}_i, \bar{\theta}_j) = d(\theta_i, \theta_j)$ for $i, j \in \{1, 2, 3\}$ [15].

If the distance between any two points on a geodesic triangle Δ does not exceed the distance between its corresponding pair of points on its comparison triangle $\bar{\Delta}$, then the geodesic space X is a CAT(0) space.

Let $\bar{\Delta}$ be a comparison triangle for a geodesic triangle Δ in X . The Δ is satisfy the CAT(0) inequality if $\forall \theta, \eta \in \Delta$ and its comparison points $\bar{\theta}, \bar{\eta} \in \bar{\Delta}$ such that

$$d(\theta, \eta) \leq d_{R^2}(\bar{\theta}, \bar{\eta}).$$

A complete CAT(0) space is often called Hadamard space [10].

Let θ, η, ζ are points of X and η_0 be the midpoint of the segment $[\eta, \zeta]$, denoted by $\frac{\eta \oplus \zeta}{2}$, then the CAT(0) inequality gives

$$(1.1) \quad d^2(\theta, \eta_0) \leq \frac{1}{2}d^2(\theta, \eta) + \frac{1}{2}d^2(\theta, \zeta) - \frac{1}{4}d^2(\eta, \zeta).$$

This is called the (CN) inequality of Bruhat and Tits [2].

A geodesic space is said to be CAT(0) space if and only if it satisfies the (CN) inequality [15].

Fixed point theory in CAT(0) spaces was first studied by [28]. He proved that every nonexpansive mapping defined on a closed, bounded convex subset of a complete CAT(0) space always had a fixed point.

Mann [30] proposed the iteration process as follows:

$$(1.2) \quad \begin{cases} \theta_1 \in D \\ \theta_{n+1} = (1 - \sigma_n)\theta_n \oplus \sigma_n\Omega\theta_n, \forall n \in N. \end{cases}$$

where $\{\sigma_n\}$ is a sequence such that $0 \leq \sigma_n \leq 1$, for all $n \in N$.

Ishikawa iteration [24] proposed the iteration process as follows:

$$(1.3) \quad \begin{cases} \theta_1 \in D \\ \theta_{n+1} = (1 - \sigma_n)\theta_n \oplus \sigma_n\Omega\eta_n, \\ \eta_n = (1 - \tau_n)\theta_n \oplus \tau_n\Omega\theta_n, n \in N, \end{cases}$$

where $\{\sigma_n\}$ and $\{\tau_n\}$ are sequences such that $0 \leq \sigma_n, \tau_n \leq 1$, for all $n \in N$.

Agarwal et al. [23] proposed the iteration process as follows:

$$(1.4) \quad \begin{cases} \theta_1 \in D \\ \theta_{n+1} = (1 - \sigma_n)\Omega\theta_n \oplus \sigma_n\Omega\eta_n, \\ \eta_n = (1 - \tau_n)\theta_n \oplus \tau_n\Omega\theta_n, n \in N, \end{cases}$$

where $\{\sigma_n\}$ and $\{\tau_n\}$ are sequences such that $0 \leq \sigma_n, \tau_n \leq 1$, for all $n \in N$.

In 2016, Thakur et al. [3] proposed the iteration process as follows:

$$(1.5) \quad \begin{cases} \theta_1 \in D \\ \theta_{n+1} = \Omega\eta_n, \\ \eta_n = \Omega((1 - \tau_n)\theta_n \oplus \tau_n\Omega\zeta_n), \\ \zeta_n = (1 - \delta_n)\theta_n \oplus \delta_n\Omega\theta_n, n \in N, \end{cases}$$

where $\{\tau_n\}$ and $\{\delta_n\}$ are sequences such that $0 \leq \tau_n, \delta_n \leq 1$, for all $n \in N$.

In 2020, Ullah et al. [11] proposed the iteration process, which they call it 'K' iteration process, as follows:

$$(1.6) \quad \begin{cases} \theta_1 \in D \\ \theta_{n+1} = \Omega\eta_n, \\ \eta_n = \Omega((1 - \tau_n)\Omega\theta_n \oplus \tau_n\Omega\zeta_n), \\ \zeta_n = (1 - \delta_n)\theta_n \oplus \delta_n\Omega\theta_n, n \in N, \end{cases}$$

where $\{\tau_n\}$ and $\{\delta_n\}$ are sequences such that $0 \leq \tau_n, \delta_n \leq 1$, for all $n \in N$.

In 2023, Samir et al. [25], introduced an iteration process, as follows;

$$(1.7) \quad \begin{cases} \theta_1 \in D \\ \theta_{n+1} = \Omega((1 - \sigma_n)\Omega\eta_n \oplus \sigma_n\Omega\eta_n), \\ \eta_n = \Omega((1 - \tau_n)\zeta_n \oplus \tau_n\Omega\zeta_n), \\ \zeta_n = \Omega((1 - \delta_n)\theta_n \oplus \delta_n\Omega\theta_n), n \in N, \end{cases}$$

where the sequences $\{\sigma_n\}, \{\tau_n\}$ and $\{\delta_n\}$ are such that $0 \leq \sigma_n, \tau_n, \delta_n \leq 1$, for all $n \geq 1$. They also show by numerical experiment the iterative algorithm (1.7) is better compared [25] to some iterative algorithms namely Ishikawa iteration, Agrawal iteration, Thakur iteration, and 'K' iteration.

Consider D be a nonempty closed, convex subset of a complete CAT(0) space X and $\Omega: D \rightarrow D$ be a mapping with $S_f(\Omega) \neq \emptyset$. Suppose, we define a sequence $\{\theta_n\}$, generated iteratively by

$$(1.8) \quad \begin{cases} \theta_1 \in D \\ \theta_{n+1} = \Omega^n((1 - \sigma_n)\Omega^n\eta_n \oplus \sigma_n\Omega^n\eta_n), \\ \eta_n = \Omega^n((1 - \tau_n)\zeta_n \oplus \tau_n\Omega^n\zeta_n), \\ \zeta_n = \Omega^n((1 - \delta_n)\theta_n \oplus \delta_n\Omega^n\theta_n), n \in N, \end{cases}$$

Note: If we take $\Omega^n = \Omega$ in (1.8), then this iteration becomes (1.7).

2. PRELIMINARIES

Let us recall some known lemmas and definitions in the existing literature on this concept, which are to be used later to prove the main results of this study.

Definition 2.1. Let (X, d) be a metric space and D It's a nonempty subset. Let $\Omega: D \rightarrow D$ be a mapping. A point $\theta \in D$ is called a fixed point of Ω if $\Omega\theta = \theta$. We will also denote by $S_f(\Omega)$ the set of fixed points of Ω , that is, $S_f(\Omega) = \{\theta \in D: \Omega\theta = \theta\}$.

Definition 2.2. Let (X, d) be a CAT(0) space and D be its nonempty subset of X . Then $\Omega: D \rightarrow D$ is said to be

1. contraction, if $d(\Omega\theta, \Omega\eta) \leq \alpha d(\theta, \eta)$, for all $\theta, \eta \in D, 0 \leq \alpha < 1$;
2. nonexpansive, if $d(\Omega\theta, \Omega\eta) \leq d(\theta, \eta)$, for all $\theta, \eta \in D$;
3. quasi-nonexpansive, if $d(\Omega\theta, p) \leq d(\theta, p)$, for all $\theta \in D, p \in S_f(T)$;
4. asymptotically quasi-nonexpansive, if there exists a sequence $\{u_n\} \in [0, \infty)$ with the property $\lim_{n \rightarrow \infty} u_n = 0$ and such that $d(\Omega^n\theta, p) \leq (1 + u_n)d(\theta, p)$, for all $\theta \in D, p \in S_f(\Omega)$;

5. semi-compact if for a sequence $\{\theta_n\}$ in D with $\lim_{n \rightarrow \infty} d(\theta_n, \Omega\theta_n) = 0$, there exists a subsequence $\theta_{n_k} \rightarrow \theta \in D$.

Lemma 2.1. Let X be a CAT(0) space.

(a) Let $\theta, \eta \in X$, for each $t \in [0,1]$, there exists a unique point $\zeta \in [\theta, \eta]$ such that [22]

$$(2.1) \quad d(\theta, \zeta) = td(\theta, \eta), \quad d(\eta, \zeta) = (1-t)d(\theta, \eta).$$

We use the notation $(1-t)\theta \oplus t\eta$ for unique point ζ , satisfying (2.1).

(b) For all $t \in [0,1]$, and $\theta, \eta, \zeta \in X$ such that [22]

$$(2.2) \quad d\left(\left((1-t)\theta \oplus t\eta\right), \zeta\right) \leq (1-t)d(\theta, \zeta) + td(\eta, \zeta).$$

(c) For $\theta, \eta, \zeta \in X$, and $t \in [0,1]$ such that [22]

$$(2.3) \quad d^2\left(\left((1-t)\theta \oplus t\eta\right), \zeta\right) \leq (1-t)d^2(\theta, \zeta) + td^2(\eta, \zeta) - t(1-t)d^2(\theta, \eta).$$

Lemma 2.2. [9] Let $\{\theta_n\}$ and $\{\eta_n\}$ be two sequences of positive real numbers satisfying

$$\theta_{n+1} \leq (1 + \eta_n)\theta_n, \quad \forall n \in N.$$

If $\sum_{n=1}^{\infty} \eta_n < \infty$, then $\sum_{n=1}^{\infty} \theta_n < \infty$.

Lemma 2.3. [1] Suppose $\{\theta_n\}$ and $\{\eta_n\}$ be two non-negative sequences of real numbers such that

$$\theta_{n+1} \leq \theta_n + \eta_n, \quad \forall n \in N.$$

If $\sum_{n=1}^{\infty} \eta_n < \infty$, then $\lim_{n \rightarrow \infty} \theta_n$ exists.

3. MAIN RESULTS

Theorem 3.1. Let D be a nonempty closed and convex subset of a complete CAT(0) space X . Let $\Omega: D \rightarrow D$ be an asymptotically quasi-nonexpansive mapping with $S_f(\Omega) \neq \emptyset$ and $\{u_n\}$ be a nonnegative sequence of real numbers with $\sum_{n=1}^{\infty} u_n < \infty$. Let sequence $\{\theta_n\}$ is defined by the iteration process (1.8). Then the sequence $\{\theta_n\}$ converges strongly to a fixed point of Ω if $\liminf_{n \rightarrow \infty} d(\theta_n, S_f(\Omega)) = 0$ or $\limsup_{n \rightarrow \infty} d(\theta_n, S_f(\Omega)) = 0$, where $d(\theta_n, S_f(\Omega)) = \inf_{\zeta \in S_f(\Omega)} d(\theta, \zeta)$.

Proof: Let $p \in S_f(\Omega)$. Since Ω is an asymptotically quasi-nonexpansive mapping, there exists a sequence $\{u_n\} \in [0, \infty)$ with the property $\lim_{n \rightarrow \infty} u_n = 0$ and such that $d(\Omega^n\theta, p) \leq (1 + u_n)d(\theta, p)$, for all $\theta \in D, p \in S_f(\Omega)$.

By combining this inequality and Lemma 2.1, we get

$$\begin{aligned} d(\zeta_n, p) &= d(\Omega^n((1 - \delta_n)\theta_n \oplus \delta_n\Omega^n\theta_n), p) \\ &\leq (1 + u_n)d\left(\left((1 - \delta_n)\theta_n \oplus \delta_n\Omega^n\theta_n\right), p\right) \\ &\leq (1 + u_n)[(1 - \delta_n)d(\theta_n, p) + \delta_n d(\Omega^n\theta_n, p)] \\ &\leq (1 + u_n)[(1 - \delta_n)d(\theta_n, p) + \delta_n(1 + u_n)d(\theta_n, p)], \\ &= (1 + u_n)(1 + \delta_n u_n)d(\theta_n, p), \end{aligned}$$

this gives

$$(3.1) \quad d(\zeta_n, p) \leq (1 + u_n)(1 + \delta_n u_n)d(\theta_n, p).$$

$$\begin{aligned} d(\eta_n, p) &= d(\Omega^n((1 - \tau_n)\zeta_n \oplus \tau_n\Omega^n\zeta_n), p) \\ &\leq (1 + u_n)d\left(\left((1 - \tau_n)\zeta_n \oplus \tau_n\Omega^n\zeta_n\right), p\right) \\ &\leq (1 + u_n)[(1 - \tau_n)d(\zeta_n, p) + \tau_n d(\Omega^n\zeta_n, p)] \\ &\leq (1 + u_n)[(1 - \tau_n)d(\zeta_n, p) + \tau_n(1 + u_n)d(\zeta_n, p)] \\ &\leq (1 + u_n)(1 + \tau_n u_n)d(\zeta_n, p) \end{aligned}$$

this gives

$$(3.2) \quad d(\eta_n, p) \leq (1 + u_n)(1 + \tau_n u_n)d(\zeta_n, p).$$

From (3.1) and (3.2), we have,

$$\begin{aligned} d(\eta_n, p) &\leq (1 + u_n)(1 + \tau_n u_n)d(\zeta_n, p) \\ &\leq (1 + u_n)^2(1 + \tau_n u_n)(1 + \delta_n u_n)d(\theta_n, p) \\ &\leq (1 + u_n)^4 d(\theta_n, p), \end{aligned}$$

this gives

$$\begin{aligned} (3.3) \quad d(\eta_n, p) &\leq (1 + u_n)^4 d(\theta_n, p) \\ d(\theta_{n+1}, p) &= d(\Omega((1 - \sigma_n)\Omega^n\eta_n \oplus \sigma_n\Omega^n\eta_n), p) \\ &\leq (1 + u_n)d\left(\left((1 - \sigma_n)\Omega^n\eta_n \oplus \sigma_n\Omega^n\eta_n\right), p\right) \\ &\leq (1 + u_n)[(1 - \sigma_n)(1 + u_n)d(\eta_n, p) + \sigma_n(1 + u_n)d(\eta_n, p)] \\ &\leq (1 + u_n)^2 d(\eta_n, p) \end{aligned}$$

$$\leq (1 + u_n)^6 d(\theta_n, p),$$

this gives,

$$(3.4) \quad d(\theta_{n+1}, p) \leq (1 + u_n)^6 d(\theta_n, p).$$

When $\theta \geq 0$ and $1 + \theta \leq e^\theta$, we have $(1 + \theta)^6 \leq e^{6\theta}$. Thus,

$$\begin{aligned} d(\theta_{n+m}, p) &\leq (1 + u_{n+m-1})^6 d(\theta_{n+m-1}, p) \\ &\leq e^{6u_{n+m-1}} d(\theta_{n+m-1}, p) \\ &\leq \dots \\ &\leq e^{6\sum_{k=n}^{n+m-1} u_k} d(\theta_n, p). \end{aligned}$$

Let $e^{6\sum_{k=n}^{n+m-1} u_k} = K$. Thus, there exists a constant $K > 0$ such that

$$d(\theta_{n+m}, p) \leq Kd(\theta_n, p).$$

This gives,

$$\begin{aligned} d(\theta_{n+1}, S_f(\Omega)) &\leq (1 + u_n)^6 d(\theta_n, S_f(\Omega)) \\ &= (1 + 6u_n + 15u_n^2 + 20u_n^3 + 15u_n^4 + 6u_n^5 + u_n^6) d(\theta_n, S_f(\Omega)). \end{aligned}$$

Since sequence $\{u_n\}$ is nonnegative and $\sum_{n=1}^\infty u_n < \infty$, we have

$$\sum_{n=1}^\infty (6u_n + 15u_n^2 + 20u_n^3 + 15u_n^4 + 6u_n^5 + u_n^6) < \infty.$$

Lemma 2.2 and $\liminf_{n \rightarrow \infty} d(\theta_n, S_f(\Omega)) = 0$ or $\limsup_{n \rightarrow \infty} d(\theta_n, S_f(\Omega)) = 0$, gives that

$$(3.5) \quad \lim_{n \rightarrow \infty} d(\theta_n, S_f(\Omega)) = 0.$$

Now, we have to show that $\{\theta_n\}$ is Cauchy sequence in D . Since $\lim_{n \rightarrow \infty} d(\theta_n, S_f(\Omega)) = 0$, for each $\epsilon > 0$, $p' \in S_f(\Omega)$, there $\exists n' \in N$ such that

$$(3.6) \quad d(\theta_n, p') < \frac{\epsilon}{K+1}, \forall n > n'.$$

$$(3.7) \quad d(\theta_n, S_f(\Omega)) < \frac{\epsilon}{K+1}, \forall n > n'.$$

Thus, there $\exists p' \in S_f(\Omega)$ such that

We obtain that

$$\begin{aligned} d(\theta_{n+m}, \theta_n) &\leq d(\theta_{n+m}, p') + d(p', \theta_n) \\ &\leq Kd(\theta_n, p') + d(\theta_n, p') \\ &\leq Kd(\theta_n, p') + d(\theta_n, p') \\ &\leq (K + 1)d(\theta_n, p'), \\ &\leq (K + 1) \frac{1}{K+1} = \epsilon \forall m, n > n'. \end{aligned}$$

Therefore, $\{\theta_n\}$ is Cauchy sequence in D . Since the set D is complete, the sequence $\{\theta_n\}$ must be converges to a point of D .

Let $\lim_{n \rightarrow \infty} \theta_n = p \in D$. Now, we shall prove that $p \in S_f(\Omega)$. Since $\lim_{n \rightarrow \infty} \theta_n = p, \forall \epsilon' > 0$, there $\exists n'' \in N$ such that

$$(3.8) \quad d(\theta_n, p) < \frac{\epsilon'}{2(2+u_1)} \forall n > n''.$$

From (3.8), for each $\epsilon' > 0$, there $\exists n''' \in N$ such that

$$d(\theta_n, S_f(\Omega)) < \frac{\epsilon'}{2(4+3u_1)}, \forall n > n'''.$$

In particular, $\inf\{d(\theta_{n''''}, p) : p \in S_f(\Omega)\} < \frac{\epsilon'}{2(4+3u_1)}$. Thus, there must exist $p_1 \in S_f(\Omega)$ such that

$$(3.9) \quad d(\theta_{n''''}, p_1) < \frac{\epsilon'}{2(4+3u_1)} \forall n > n'''.$$

From (3.8) and (3.9), we have

$$\begin{aligned} d(\Omega p, p) &\leq d(\Omega p, p_1) + d(p_1, \Omega \theta_{n''''}) + d(\Omega \theta_{n''''}, p_1) + d(p_1, \theta_{n''''}) + d(\theta_{n''''}, p) \\ &\leq d(\Omega p, p_1) + 2d(\Omega \theta_{n''''}, p_1) + d(p_1, \theta_{n''''}) + d(\theta_{n''''}, p) \\ &\leq (1 + u_1)d(p, p_1) + 2(1 + u_1)d(\theta_{n''''}, p_1) + d(p_1, \theta_{n''''}) + d(\theta_{n''''}, p) \\ &\leq (1 + u_1)d(p, \theta_{n''''}) + (1 + u_1)d(\theta_{n''''}, p_1) + 2(1 + u_1)d(\theta_{n''''}, p_1) + d(p_1, \theta_{n''''}) + d(\theta_{n''''}, p) \\ &\leq (1 + u_1) \frac{\epsilon'}{2(2+u_1)} + (1 + u_1) \frac{\epsilon'}{2(4+3u_1)} + 2(1 + u_1) \frac{\epsilon'}{2(4+3u_1)} + \frac{\epsilon'}{2(4+3u_1)} + \frac{\epsilon'}{2(2+u_1)} \\ &= (2 + u_1) \frac{\epsilon'}{2(2+u_1)} + (4 + 3u_1) \frac{\epsilon'}{2(4+3u_1)} \\ &= (2 + u_1) \frac{\epsilon'}{2(2+u_1)} + (4 + 3u_1) \frac{\epsilon'}{2(4+3u_1)} \\ &= \frac{\epsilon'}{2} + \frac{\epsilon'}{2} = \epsilon' \end{aligned}$$

Since ϵ' is arbitrary, so $d(\Omega p, p) = 0$, i.e., $\Omega p = p$. Therefore, $p \in S_f(\Omega)$.

Definition 3.1. Let D be a nonempty subset of a CAT(0) space X . Let $\Omega: D \rightarrow D$ be a mapping with $S_f(\Omega) \neq \emptyset$ are said to satisfy the condition (A) if there exists a nondecreasing function $g: [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$, $g(r) > 0, \forall r \in (0, \infty)$ such that

$$d(\theta, \Omega\theta) \geq g\left(d\left(\theta, S_f(\Omega)\right)\right), \forall \theta \in D,$$

where $d\left(\theta, S_f(\Omega)\right) = \inf_{t \in S_f(\Omega)} d(\theta, t)$.

Theorem 3.2. Let D be a nonempty bounded, closed, and convex subset of a complete CAT(0) space X . Let $\Omega: D \rightarrow D$ be a mapping satisfying condition (A) with a nonempty, closed set $S_f(\Omega)$. Let sequence $\{\theta_n\}$ is defined by the iteration process (1.8). If Ω satisfying condition (A) and $\lim_{n \rightarrow \infty} d(\theta_n, \Omega\theta_n) = 0$ then $\{\theta_n\}$ converges strongly to a point $p \in S_f(\Omega)$.

Proof: Since Ω satisfying condition (A) so there \exists a non-decreasing function $g: [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ and $g(r) > 0 \forall r > 0$ such that

$$d(\theta, \Omega\theta) \geq g\left(d\left(\theta, S_f(\Omega)\right)\right) \forall \theta \in D,$$

where $d\left(\theta, S_f(\Omega)\right) = \inf_{t \in S_f(\Omega)} d(\theta, t)$.

This gives

$$\lim_{n \rightarrow \infty} g\left(d\left(\theta_n, S_f(\Omega)\right)\right) \leq \lim_{n \rightarrow \infty} d(\theta_n, \Omega\theta_n) = 0,$$

we get $\lim_{n \rightarrow \infty} g\left(d\left(\theta_n, S_f(\Omega)\right)\right) = 0$. Since g is a non-decreasing function with $g(0) = 0$ it follows that $\lim_{n \rightarrow \infty} d\left(\theta_n, S_f(\Omega)\right) = 0$. This implies, that every $\epsilon > 0$, there exists a positive integer n_0 such that

$$d\left(\theta_n, S_f(\Omega)\right) < \frac{\epsilon}{4}, \forall n \geq n_0$$

This implies, $\inf\{d(\theta_{n_0}, p) : p \in S_f(\Omega)\} < \frac{\epsilon}{4}$, so there will be $p^* \in S_f(\Omega)$ such that $d(\theta_{n_0}, p^*) < \frac{\epsilon}{2}$.

Moreover, for all $m, n \geq n_0$, we have

$$\begin{aligned} d(\theta_m, \theta_n) &< d(\theta_m, p^*) + d(p^*, \theta_n) \\ &\leq 2d(\theta_{n_0}, p^*) \\ &< 2 \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This shows that $\{\theta_n\}$ is a Cauchy sequence in the closed subset D of the complete CAT(0) space, which shows that $\{\theta_n\}$ must converge to a point $p \in D$. We obtain $d\left(\theta, S_f(\Omega)\right) = 0$ because $\lim_{n \rightarrow \infty} d\left(\theta_n, S_f(\Omega)\right) = 0$. Using the fact that $S_f(\Omega)$ is closed, the conclusion is that $p \in S_f(\Omega)$.

Theorem 3.3. Let D be a nonempty closed and convex subset of a complete CAT(0) space X . Let $\Omega: D \rightarrow D$ be asymptotically quasi-nonexpansive mapping with $S_f(\Omega) \neq \emptyset$ and $\{u_n\}$ be a nonnegative sequence of real numbers with $\sum_{n=1}^{\infty} u_n < \infty$. Let sequence $\{\theta_n\}$ is defined by the iteration process (1.8). Then the sequence $\{\theta_n\}$ converges strongly to a fixed point of Ω if Ω is semi-compact and $\lim_{n \rightarrow \infty} d(\theta_n, \Omega\theta_n) = 0$.

Proof: By the hypothesis, we have $\lim_{n \rightarrow \infty} d(\theta_n, \Omega\theta_n) = 0$. Since Ω is semi-compact so there \exists a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\lim_{n \rightarrow \infty} \theta_{n_k} = p \in D$. Hence,

$$d(p, \Omega p) = \lim_{n \rightarrow \infty} d(\theta_{n_k}, \Omega\theta_{n_k}) = 0.$$

Thus, $p \in S_f(\Omega)$. By (3.4),

$$\begin{aligned} d(\theta_{n+1}, p) &\leq (1 + u_n)^6 d(\theta_n, p) \\ &= (1 + 6u_n + 15u_n^2 + 20u_n^3 + 15u_n^4 + 6u_n^5 + u_n^6) d(\theta_n, p). \end{aligned}$$

Since sequence $\{u_n\}$ is nonnegative and $\sum_{n=1}^{\infty} u_n < \infty$, we have

$$\sum_{n=1}^{\infty} (6u_n + 15u_n^2 + 20u_n^3 + 15u_n^4 + 6u_n^5 + u_n^6) < \infty.$$

By Lemma (2.2), $\lim_{n \rightarrow \infty} d(\theta_n, p)$ exists and $\lim_{n \rightarrow \infty} \theta_{n_k} = p \in S_f(\Omega)$, gives that

$$\lim_{n \rightarrow \infty} \theta_n = p.$$

4. EXAMPLES :

One notes that quasi-nonexpansive mapping $\Omega: D \rightarrow D$ is an asymptotically quasi-nonexpansive; but there exists nonlinear asymptotically quasi-nonexpansive mapping which is not quasi-nonexpansive.

Example 4.1. (An asymptotically quasi-nonexpansive type mapping which is not quasi-nonexpansive)

Let a function $\Omega: [0,1] \rightarrow R$ such that

$$\Omega\theta = \theta^{\frac{1}{2}}$$

Here $S_f(\Omega) = \{0,1\}$ = The set of fixed points of Ω .

For $p \in S_f(\Omega)$ and $n \in N$, we have

$$|\Omega^n\theta - p| = \left| \theta^{\frac{1}{2^n}} - p \right| \leq \left(1 + \frac{1}{n} \right) |\theta - p|,$$

this shows that Ω is an asymptotically quasi-nonexpansive.

If we take $\theta = 0.9, n = 2$, we have $|\Omega^2(0.9) - 0| \leq \left| (0.9)^{\frac{1}{4}} - 0 \right| = 0.974$ (up to three decimal) $> |0.9 - 0| = 0.9$, this implies that Ω is not quasi-nonexpansive.

Example 4.2. (An asymptotically quasi-nonexpansive type mapping whose fixed point set is not closed)

Let a mapping $\Omega: [0,1] \rightarrow R$ defined as

$$\Omega\theta = \begin{cases} \theta, & \text{if } \theta \in [0,1] \cap Q \\ \theta^2, & \text{otherwise.} \end{cases}$$

Notice that the set of fixed points of the mapping $\Omega = S_f(\Omega) = [0,1] \cap Q$, which is not closed in R .

5. NUMERICAL EXPERIMENT:

Let $X = R$ be the real line with the usual metric space d and $D = [0,10]$. Define a mapping $\Omega: D \rightarrow D$ such that $\Omega(\theta) = \frac{\theta}{2}$. Then we shall show that Ω is asymptotically quasi-nonexpansive.

Here $S_f(\Omega) = \{0\}$ = The set of fixed points of Ω . If we take a sequence $\{\mu_n\} = \{1 + \frac{1}{n}\}$,

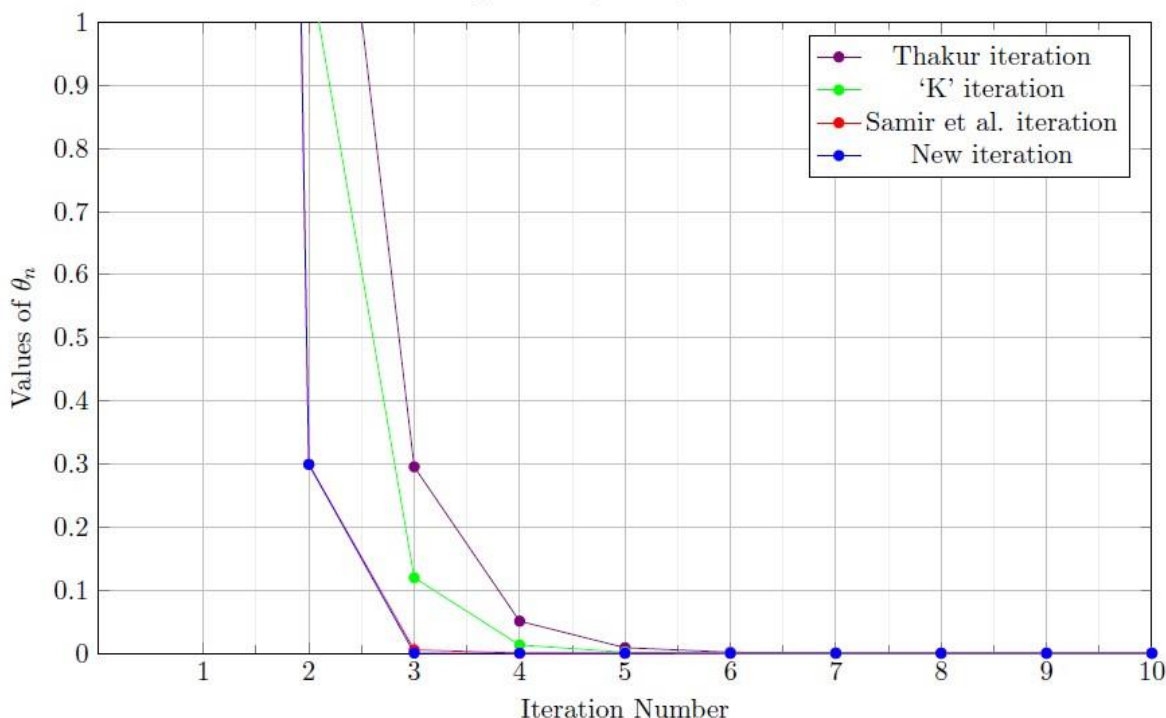
$$|\Omega^n(\theta) - 0| = \left| \frac{\theta}{2^n} - 0 \right| \leq \left(1 + \frac{1}{n} \right) |\theta - 0|.$$

This shows that Ω is asymptotically quasi-nonexpansive mapping.

We obtained the influence of the initial point for the new iteration scheme (1.8) by $\sigma_n = \tau_n = \delta_n = \frac{1}{2}$ and $\theta_1 = 10$ in Table 1.

No	Thakur iteration	'K' Iteration	Samir et al. Iteration	New Iteration
$n = 1$	10.000000000000000	10.000000000000000	10.000000000000000	10.000000000000000
$n = 2$	1.718750000000000	1.093750000000000	0.299377441406250	0.299377441406250
$n = 3$	0.295410156250000	0.119628906250000	0.005373269463291	0.000228824218979
$n = 4$	0.050773620605469	0.013084411621094	0.000094487687007	0.000000008838064
$n = 5$	0.008726716041565	0.001431107521057	0.000001660927408	0.000000000000019
$n = 6$	0.001499904319644	0.000156527385116	0.000000029195993	0.000000000000000
$n = 7$	0.000257796054939	0.000017120182747	0.000000000513211	0.000000000000000
$n = 8$	0.000044308696943	0.000001872519988	0.000000000009021	0.000000000000000
$n = 9$	0.000007615557287	0.000000204806874	0.000000000000159	0.000000000000000
$n = 10$	0.000001308923909	0.000000022400752	0.000000000000003	0.000000000000000
$n = 11$	0.000000224971297	0.000000002450082	0.000000000000000	0.000000000000000
$n = 12$	0.000000038666942	0.000000000267978	0.000000000000000	0.000000000000000
$n = 13$	0.000000006645881	0.000000000029310	0.000000000000000	0.000000000000000
$n = 14$	0.000000001142261	0.000000000003206	0.000000000000000	0.000000000000000
$n = 15$	0.000000000196326	0.000000000000351	0.000000000000000	0.000000000000000
$n = 16$	0.000000000033744	0.000000000000038	0.000000000000000	0.000000000000000
$n = 17$	0.000000000005800	0.000000000000004	0.000000000000000	0.000000000000000
$n = 18$	0.000000000000997	0.000000000000000	0.000000000000000	0.000000000000000
$n = 19$	0.000000000000171	0.000000000000000	0.000000000000000	0.000000000000000
$n = 20$	0.000000000000029	0.000000000000000	0.000000000000000	0.000000000000000
$n = 21$	0.000000000000005	0.000000000000000	0.000000000000000	0.000000000000000

Graph Corresponding to Table 1



6. CONCLUSIONS:

In this work, we have given some fixed point results for an asymptotically quasi-nonexpansive mapping and also proposed a three-step new iteration process for the approximation fixed point of the class of mappings in the frame of CAT(0) spaces. Our example 4.1 shows that every non-linear asymptotically quasi-nonexpansive mapping need not be quasi-nonexpansive. Our example 4.2 shows that the set of fixed points of asymptotically quasi-nonexpansive mappings need not be closed. Our numerical experiment shows that our new iteration scheme (1.8) is better compared to some existing iterative schemes in the literature.

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