Common Fixed Point Results For Quadruple Transformations In Vector Valued Rectangular Metric Spaces

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Abstract: In this manuscript, we consider vector valued Rectangular metric space in which metric is Riesz space valued. We establish the existence of common fixed point results for four self-maps in vector valued rectangular metric space. Established results extend and generalize several fixed point results for scalar valued case in the literature.

Keywords: Riesz space, Vector valued rectangular metric space, Weakly compatible.

1. INTRODUCTION AND PRELIMINARIES

Let F_1 and F_2 be two self-maps on a metric space (W, κ). If there exists a point $\varsigma \in W$ such that $F_1\varsigma = F_2\varsigma = \varsigma$ then ς is common fixed point (denoted as c.f.p) of F_1 and F_2. The analysis of c.f.p for contraction has been at the center of vital research activity. In 1982, Sessa [17] was first who interpreted weakly commuting maps and established some c.f.p results for a couple of maps. Afterwards in 1988, Jungck [8] generalized the concept of commuting maps to compatible maps and later to weakly compatible maps [9]. Many authors then investigated unique c.f.p by using the concept of weakly compatible maps. Our result generalize well-known comparable results in the literature, see [15, 10, 11]. Now, several definitions and results that will be utilized in the sequal are discussed here. One may refer, Aliprantis and Border [1] for definitions and results regarding the Riesz space, and Cevik and Altun [2] for vector metric space. A partially ordered set is a lattice if every couple of elements has both supremum and infimum. A Riesz space Q is a partially ordered vector space which is also a lattice under its ordering. Let $\{\vartheta_n\}$ be a decreasing sequence in Q such that inf $\vartheta_n = \vartheta$, then we write $\vartheta_n \downarrow \vartheta$. If for each $\vartheta \in Q_+ = \{\varsigma \in Q : 0 \preceq \varsigma\}$ and $n \in N$ we have $\frac{1}{n}\vartheta \downarrow 0$, then Riesz space Q is called Archimedean.

Lemma 1.1. [2] Let Q be a Riesz space and $\vartheta \preceq a\vartheta$, $\forall\, \vartheta \in Q_+$ also $0 \leq a < 1$, then $\vartheta = 0$.

Definition 1.2. [2] Let W be any non-empty set and Q be a Riesz space. Then the mapping $\kappa : W \times W \to Q$ is vector metric if it possesses the properties listed below:
(a) $\kappa(\vartheta, \omega) = 0 \Leftrightarrow \vartheta = \omega$
(b) $\kappa(\vartheta, \omega) = \kappa(\omega, \vartheta)$
(c) $\kappa(\vartheta, \omega) \preceq \kappa(\vartheta, \eta) + \kappa(\eta, \omega)$, $\forall\, \eta, \vartheta, \omega \in W$
Then (W, κ, Q) is called vector metric space (denoted as v.m.s).

Example 1.3. [2] Let $Q = R^2$ and a map $\kappa : R \times R \to Q$ defined as:
$$\kappa(\vartheta, \omega) = (o_1|\vartheta - \omega|, o_2|\vartheta - \omega|)$$
where $0 < o_1 + o_2$ and $0 \leq o_1, o_2$. Then the triplet (R, κ, Q) is v.m.s.

Definition 1.4. [2] A sequence $\{\varsigma_m\}$ in a v.m.s (W, κ, Q) is called vectorial convergent (Q-convergent) to some $\varsigma \in W$, written as $\varsigma_m \overset{\kappa,Q}{\longrightarrow} \varsigma$, if there exists a sequence b_m in Q such that $b_m \downarrow 0$ as well as $\kappa(\varsigma_m, \varsigma) \preceq b_m\, \forall\, m$.

Definition 1.5. [2] A sequence $\{\varsigma_m\}$ in a v.m.s. (W,κ,Q) is called Q-Cauchy if we get a sequence b_m in Q satisfying $b_m \downarrow 0$ as well as $\kappa(\varsigma_m, \varsigma_{m+p}) \leq b_m \, \forall \, m,p$.

Definition 1.6. [2] A v.m.s (W,κ,Q) is said to be Q-complete if every Q-Cauchy sequence in W is Q-convergent to a limit in W.
The following definition was given by Branciari [4].

Definition 1.7. [4] Let W be any non-empty set and the mapping $\kappa : W \times W \to R$ s.t. for all $\vartheta, \omega \in W$ and for all distinct $\eta, \varsigma \in W$ s.t. $\eta, \varsigma \notin \{\vartheta, \omega\}$ is called rectangular metric if it possesses the properties listed below:
(a) $\kappa(\vartheta,\omega) = 0 \Leftrightarrow \vartheta = \omega$
(b) $\kappa(\vartheta,\omega) = \kappa(\omega,\vartheta)$
(c) $\kappa(\vartheta,\omega) \leq \kappa(\vartheta,\eta) + \kappa(\eta,\varsigma) + \kappa(\varsigma,\omega)$.
Then the pair (W,κ) is called rectangular metric space (r.m.s).

An example of r.m.s that isn't a metric space is provided below.
Example 1.8. [4] Let $W = R$, $0 < t \in W$ and define $\kappa : W \times W \to R$ s.t. for all $\vartheta, \omega \in W$, $\kappa(\vartheta,\omega) = \kappa(\omega,\vartheta)$ and

$$\kappa(\vartheta,\omega) = \begin{cases} 3t & \text{if } \vartheta, \omega \in \{1,2\} \text{ and } \vartheta \neq \omega, \\ 0 & \text{if } \vartheta = \omega, \\ t & \text{if } \vartheta, \omega \notin \{1,2\} \text{ and } \vartheta \neq \omega. \end{cases}$$

The topology of r.m.s need not be Hausdorff, neither the respective space is continuous. However, Sarma et al. [16] presented several possible properties of r.m.s by the following example:

Example 1.9. [16] Let $A = \{0,2\}$, $B = \{\frac{1}{n} : n \in N\}$, $W = A \cup B$. Define mapping $\kappa : W \times W \to R^+$ as follows:

$$\kappa(\vartheta,\omega) = \begin{cases} 0 & \text{if } \vartheta = \omega, \\ 1 & \text{if } \vartheta \neq \omega \text{ and } \{\vartheta,\omega\} \subseteq A \text{ or } \{\vartheta,\omega\} \subseteq B, \\ \omega & \text{if } \vartheta \in A, \omega \in B, \\ \vartheta & \text{if } \vartheta \in B, \omega \in A. \end{cases}$$

Clearly, (W,κ) is complete r.m.s. However, it was shown by Sarma et al. [16] that:
(a) In r.m.s (W,κ), there is a convergent sequence that is not Cauchy ;
(b) In r.m.s (W,κ), there is a sequence which converging to two distinct points;
(c) The intersection of two open balls with positive radius r, center at 0 and 2 is always nonempty;
(d) $\lim_{n\to\infty} \frac{1}{n}$ but $\lim_{n\to\infty} \left(\frac{1}{n}, \frac{1}{2}\right) \neq \kappa\left(0, \frac{1}{2}\right)$,
hence κ is not continuous function.
Inspired by the work of Branciari [4], we define the Riesz valued rectangular metric space as follows:

Definition 1.10. Let W be any non-empty set and Q be a Riesz space. A vector valued rectangular metric is a mapping $\kappa : W \times W \to Q$ if for all $\vartheta, \omega \in W$ and for all distinct $\eta, \varsigma \in W$ s.t. $\eta, \varsigma \notin \{\vartheta, \omega\}$, it satisfies the following properties:
(a) $\kappa(\vartheta,\omega) = 0 \Leftrightarrow \vartheta = \omega$
(b) $\kappa(\vartheta,\omega) = \kappa(\omega,\vartheta)$
(c) $\kappa(\vartheta,\omega) \leq \kappa(\vartheta,\eta) + \kappa(\eta,\varsigma) + \kappa(\varsigma,\omega)$.
And the triplet (W,κ,Q) is vector valued r.m.s.

Next, we provide an example of vactor valued r.m.s that is not a v.m.s.
Example 1.11. [12] Let $Q = R^2$, $W = \{0,1,2,3\}$ and define $\kappa : W \times W \to Q$ s.t. for all $\vartheta, \omega \in W$, $\kappa(\vartheta,\omega) = \kappa(\omega,\vartheta)$ and

$$\kappa(\vartheta,\omega) = \begin{cases} (0,0) & \text{if } \vartheta = \omega, \\ (3,3) & \text{if } \vartheta, \omega \in \{2,3\} \text{ and } \vartheta \neq \omega, \\ (1,1) & \text{if } \vartheta, \omega \notin \{2,3\} \text{ and } \vartheta \neq \omega. \end{cases}$$

Lemma 1.12. Let (W, κ, Q) be a vector valued r.m.s and $\{\vartheta_{2m}\}$ be any subsequence of a Q-Cauchy sequence $\{\vartheta_m\}$ in W such that $\vartheta_{2m} \overset{\kappa,Q}{\longrightarrow} \vartheta$. Then entire Q-Cauchy sequence $\{\vartheta_m\}$ is convergent i.e. $\vartheta_m \overset{\kappa,Q}{\longrightarrow} \vartheta$.
Proof. Since ϑ_m is Q-cauchy sequence, then there exist sequence $\{a_m\}$ in Q s.t. $a_m \downarrow 0$ and for all m and p, we have

$$\kappa(\vartheta_m, \vartheta_{m+p}) \preceq a_m. \qquad (1)$$

Also $\vartheta_{2m} \overset{\kappa,Q}{\longrightarrow} \vartheta$, then there exist $\{b_{2m}\}$ in Q with $b_{2m} \downarrow 0$ such that

$$\kappa(\vartheta_{2m}, \vartheta) \preceq b_{2m}. \qquad (2)$$

Now if $m > p$, we have

$$\kappa(\vartheta_{m+p}, \vartheta_{2m}) = \kappa(\vartheta_{m+p}, \vartheta_{m+m}) = \kappa(\vartheta_{m+p}, \vartheta_{m+p+m-p}) \preceq a_{m+p} \downarrow 0. \qquad (3)$$

And if $p > m$, we have

$$\kappa(\vartheta_{m+p}, \vartheta_{2m}) = \kappa(\vartheta_{m+m}, \vartheta_{m+p}) = \kappa(\vartheta_{m+m}, \vartheta_{m+m+p-m}) \preceq a_{m+m} \downarrow 0. \qquad (4)$$

For all $m \neq p$, we get

$$\kappa(\vartheta_m, \vartheta) \preceq \kappa(\vartheta_m, \vartheta_{m+p}) + \kappa(\vartheta_{m+p}, \vartheta_{2m}) + \kappa(\vartheta_{2m}, \vartheta) \text{ (by rectangular inequality)}$$
$$\preceq \{a_m + b_{2m} + \kappa(\vartheta_{m+p}, \vartheta_{2m})\} \downarrow 0 \text{ (using (1), (2), (3) and (4))}.$$

Hence $\vartheta_m \overset{\kappa,Q}{\longrightarrow} \vartheta$.
Likewise we can also prove that if $\{\vartheta_{2m+1}\}$ subsequence of a Q-Cauchy sequence is Q-convergent then entire Q-Cauchy sequence is conergent.

We observed with the help of an example 1.9., that a sequence in a r.m.s may converge to two distinct points. Nevertheless, there is an exceptional case in which this is not possible and this will be quite beneficial for some proofs. We generalize the following lemma of Arshad et al. [3] on vector valued r.m.s.

Lemma 1.13. Let (W, κ, Q) be a vector valued r.m.s and $\{\vartheta_m\}$ be a Q-Cauchy sequence in W such that $\vartheta_m \neq \vartheta_n$ whenever $m \neq n$. Then $\{\vartheta_m\}$ can Q-converge to atmost one point.
Proof. Let $\{\vartheta_m\}$ be a Q-Cauchy sequence in W, then there exists a_m in Q such that $a_m \downarrow 0$ and

$$\kappa(\vartheta_m, \vartheta_{m+p}) \preceq a_m. \qquad (5)$$

for all m and p. Suppose $\{\vartheta_m\}$ is Q-converge on two distinct points v^* and v, i.e. $\vartheta_m \overset{\kappa,Q}{\longrightarrow} \vartheta^*$, $\vartheta_m \overset{\kappa,Q}{\longrightarrow} \vartheta$, so there exists sequences b_m and c_m in Q such that $b_m \downarrow 0$, $c_m \downarrow 0$ and

$$\kappa(\vartheta_m, \vartheta^*) \preceq b_m, \qquad \kappa(\vartheta_m, \vartheta) \preceq c_m. \qquad (6)$$

For $m \neq p$, we have

$$\kappa(\vartheta^*, \vartheta) \preceq \kappa(\vartheta^*, \vartheta_m) + \kappa(\vartheta_m, \vartheta_{m+p}) + \kappa(\vartheta_{m+p}, \vartheta) \text{ (by rectangular inequality)}$$
$$\preceq b_m + a_m + c_{m+p} \text{ (using (5) and (6))}$$
$$\preceq a_m + b_m + c_m \ (\because c_{m+p} \preceq c_m).$$

Implise $\kappa(\vartheta^*, \vartheta) = 0$, i.e. $\vartheta^* = \vartheta$. A contradiction, hence $\{\vartheta_m\}$ can Q-converge to atmost one point.

Definition 1.14. [11] Let $F_1, F_2 : W \to W$. If $F_1\vartheta = F_2\vartheta = \varsigma$ where $\vartheta \in W$, then ς is called a point of coincidence of F_1 & F_2, and ϑ is called a coincidence point of F_1 & F_2.

Definition 1.15. [9] Two self-maps F_1 and F_2 on W are called weakly compatible (WC) if for some $\vartheta \in W$, $F_1\vartheta = F_2\vartheta$, implies $F_1F_2\vartheta = F_2F_1\vartheta$.

2. Main Results

Theorem 2.1. Let (W, κ, Q) be a vector valued r.m.s with Q-Archimedean. Suppose the self-maps F_1, F_2, F_3 & F_4 on W satisfy the following conditions:
(i) for all $\varsigma, \vartheta \in W$, and $\gamma \in [0, 1)$

$$\left.\begin{aligned} \kappa(F_1\varsigma, F_2\vartheta) &\preceq \gamma H_{\varsigma,\vartheta}(F_1, F_2, F_3, F_4) \\ \kappa(F_3\varsigma, F_2\vartheta) &\preceq \gamma H_{\varsigma,\vartheta}(F_1, F_2, F_3, F_4) \end{aligned}\right\} \qquad (7)$$

where $H_{\varsigma,\vartheta}(F_1, F_2, F_3, F_4) \in \{\kappa(F_3\varsigma, F_4\vartheta), \kappa(F_1\varsigma, F_3\varsigma), \kappa(F_2\vartheta, F_4\vartheta), \kappa(F_1\varsigma, F_4\vartheta)\}$.
(ii) $F_1(W) \subseteq F_4(W)$ and $F_2(W) \subseteq F_3(W)$.
(iii) $F_1(W), F_2(W), F_3(W)$ or $F_4(W)$ is Q-complete subspace of W.
Then the pairs $\{F_1, F_3\}$ and $\{F_2, F_4\}$ have unique point of coincidence. Moreover, if the pairs $\{F_1, F_3\}$ and $\{F_2, F_4\}$ are WC then F_1, F_2, F_3 & F_4 have c.f.p which is unique.

Proof. Let any $\varsigma_0 \in W$. Since $F_1(W) \subseteq F_4(W)$ and $F_2(W) \subseteq F_3(W)$, we can choose $\varsigma_1 \in W$ such that $F_1\varsigma_0 = F_4\varsigma_1$, $\varsigma_2 \in W$ such that $F_2\varsigma_1 = F_3\varsigma_2$. Continuing this process, for all $m \geq 0$ construct a sequence $\{\vartheta_m\}$, which is defined as:

$$\vartheta_{2m-1} = F_4\varsigma_{2m-1} = F_1\varsigma_{2m-2} \text{ and } \vartheta_{2m} = F_3\varsigma_{2m} = F_2\varsigma_{2m-1}. \quad (8)$$

Firstly, we prove that

$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) \preceq \gamma\kappa(\vartheta_{2m}, \vartheta_{2m+1}).$$

We have

$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) = \kappa(F_1\varsigma_{2m}, F_2\varsigma_{2m+1}) \preceq \gamma H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4),$$

where

$$\begin{aligned} H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) \in \{&\kappa(F_3\varsigma_{2m}, F_4\varsigma_{2m+1}), \kappa(F_1\varsigma_{2m}, F_3\varsigma_{2m}), \kappa(F_2\varsigma_{2m+1}, F_4\varsigma_{2m+1}), \\ &\kappa(F_1\varsigma_{2m}, F_4\varsigma_{2m+1})\} \\ = \{&\kappa(\vartheta_{2m}, \vartheta_{2m+1}), \kappa(\vartheta_{2m+1}, \vartheta_{2m}), \kappa(\vartheta_{2m+2}, \vartheta_{2m+1}), \\ &\kappa(\vartheta_{2m+1}, \vartheta_{2m+1})\} \\ = \{&\kappa(\vartheta_{2m}, \vartheta_{2m+1}), \kappa(\vartheta_{2m+2}, \vartheta_{2m+1}), 0\}. \end{aligned}$$

The possible three cases are:

(i) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) = \kappa(\vartheta_{2m}, \vartheta_{2m+1})$, then

$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) \preceq \gamma\kappa(\vartheta_{2m}, \vartheta_{2m+1}).$$

(ii) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) = \kappa(\vartheta_{2m+2}, \vartheta_{2m+1})$, thus, we have

$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) = 0.$$

(iii) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) = 0$, implies $\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) = 0$.

Thus for $0 \leq \gamma < 1$, we have

$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) \preceq \gamma\kappa(\vartheta_{2m}, \vartheta_{2m+1}). \quad (9)$$

Similarly, we can prove

$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+3}) \preceq \gamma\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}). \quad (10)$$

Thus, for all $m \geq 1$ and from (9) and (10),

$$\kappa(\vartheta_m, \vartheta_{m+1}) \preceq \gamma\kappa(\vartheta_{m-1}, \vartheta_m). \quad (11)$$

Repeating the process of (11), we get

$$\kappa(\vartheta_m, \vartheta_{m+1}) \preceq \gamma^m\kappa(\vartheta_0, \vartheta_1). \quad (12)$$

Now by using (7), (8), we obtain

$$\kappa(\vartheta_{2m}, \vartheta_{2m+2}) = \kappa(F_3\varsigma_{2m}, F_2\varsigma_{2m+1}) \preceq \gamma H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4),$$

where

$$\begin{aligned} H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) \in \{&\kappa(F_3\varsigma_{2m}, F_4\varsigma_{2m+1}), \kappa(F_1\varsigma_{2m}, F_3\varsigma_{2m}), \kappa(F_2\varsigma_{2m+1}, F_4\varsigma_{2m+1}), \\ &\kappa(F_1\varsigma_{2m}, F_4\varsigma_{2m+1})\} \\ = \{&\kappa(\vartheta_{2m}, \vartheta_{2m+1}), \kappa(\vartheta_{2m+1}, \vartheta_{2m}), \kappa(\vartheta_{2m+2}, \vartheta_{2m+1}), \\ &\kappa(\vartheta_{2m+1}, \vartheta_{2m+1})\} \\ = \{&\kappa(\vartheta_{2m}, \vartheta_{2m+1}), \kappa(\vartheta_{2m+2}, \vartheta_{2m+1}), 0\}. \end{aligned}$$

The possible three cases are:

(i) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) = \kappa(\vartheta_{2m}, \vartheta_{2m+1})$, then

$$\begin{aligned} \kappa(\vartheta_{2m}, \vartheta_{2m+2}) &\preceq \gamma\kappa(\vartheta_{2m}, \vartheta_{2m+1}) \\ &\preceq \gamma^{2m+1}\kappa(\vartheta_0, \vartheta_1) \text{ (by using (12)).} \end{aligned}$$

(ii) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) = \kappa(\vartheta_{2m+2}, \vartheta_{2m+1})$, then

$$\begin{aligned} \kappa(\vartheta_{2m}, \vartheta_{2m+2}) &\preceq \gamma\kappa(\vartheta_{2m+2}, \vartheta_{2m+1}) \\ &\preceq \gamma^{2m+2}\kappa(\vartheta_0, \vartheta_1) \text{ (by using (12))} \\ &\preceq \gamma^{2m+1}\kappa(\vartheta_0, \vartheta_1). \end{aligned}$$

(iii) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) = 0$, implies $\kappa(\vartheta_{2m}, \vartheta_{2m+2}) = 0$.

For all $m \geq 1$, we get

$$\kappa(\vartheta_{2m}, \vartheta_{2m+2}) \preceq \gamma^{2m+1}\kappa(\vartheta_0, \vartheta_1). \quad (13)$$

By using rectangular inequality, we have

$$\begin{aligned} \kappa(\vartheta_{2m+1}, \vartheta_{2m+3}) &\preceq \kappa(\vartheta_{2m+1}, \vartheta_{2m}) + \kappa(\vartheta_{2m}, \vartheta_{2m+2}) + \kappa(\vartheta_{2m+2}, \vartheta_{2m+3}) \\ &\preceq \gamma^{2m}\kappa(\vartheta_0, \vartheta_1) + \gamma^{2m+1}\kappa(\vartheta_0, \vartheta_1) + \gamma^{2m+2}\kappa(\vartheta_0, \vartheta_1) \\ &\preceq 3\gamma^{2m}\kappa(\vartheta_0, \vartheta_1) \ (\because\ 0 \leq \gamma < 1). \quad (14) \end{aligned}$$

Also from (13), we can write

$$\kappa(\vartheta_{2m}, \vartheta_{2m+2}) \preceq \gamma^{2m+1}\kappa(\vartheta_0, \vartheta_1) \preceq 3\gamma^{2m}\kappa(\vartheta_0, \vartheta_1). \quad (15)$$

Therefore from (14) and (15), we obtain

$$\kappa(\vartheta_m, \vartheta_{m+2}) \preceq 3\gamma^m\kappa(\vartheta_0, \vartheta_1), \quad (16)$$

for all m ≥ 1. For $\{\vartheta_m\}$, we consider $\kappa(\vartheta_m,\vartheta_{m+p})$ in two cases.

Case 1. If p is odd say $2n+1$, $n\in N\cup\{0\}$, then by rectangular inequality

$$\begin{aligned}\kappa(\vartheta_m,\vartheta_{m+p}) &\preceq \kappa(\vartheta_m,\vartheta_{m+1})+\kappa(\vartheta_{m+1},\vartheta_{m+2})+\kappa(\vartheta_{m+2},\vartheta_{m+p})\\ &\preceq \kappa(\vartheta_m,\vartheta_{m+1})+\kappa(\vartheta_{m+1},\vartheta_{m+2})+\kappa(\vartheta_{m+2},\vartheta_{m+3})+\kappa(\vartheta_{m+3},\vartheta_{m+4})+\kappa\big(\vartheta_{m+4},\vartheta_{m+p}\big)\\ &\preceq \kappa(\vartheta_m,\vartheta_{m+1})+\kappa(\vartheta_{m+1},\vartheta_{m+2})+\cdots+\kappa(\vartheta_{m+2n-1},\vartheta_{m+2n})+\kappa(\vartheta_{m+2n},\vartheta_{m+p})\\ &\preceq \gamma^m\kappa(\vartheta_0,\vartheta_1)+\gamma^{2m+1}\kappa(\vartheta_0,\vartheta_1)+\cdots+\gamma^{2m+2}\kappa(\vartheta_0,\vartheta_1) \quad \text{(by using (12))}\\ &\preceq \left\{\frac{\gamma^m}{1-\gamma}\kappa(\vartheta_0,\vartheta_1)\right\}\downarrow 0.\end{aligned}$$

Case 2. If p is even say $2n$, $n\in N$, then by rectangular inequality

$$\begin{aligned}\kappa(\vartheta_m,\vartheta_{m+p}) &\preceq \kappa(\vartheta_m,\vartheta_{m+2})+\kappa(\vartheta_{m+2},\vartheta_{m+3})+\kappa(\vartheta_{m+3},\vartheta_{m+p})\\ &\preceq \kappa(\vartheta_m,\vartheta_{m+2})+\kappa(\vartheta_{m+2},\vartheta_{m+3})+\kappa(\vartheta_{m+3},\vartheta_{m+4})+\kappa(\vartheta_{m+4},\vartheta_{m+5})+\kappa(\vartheta_{m+5},\vartheta_{m+p})\\ &\preceq \kappa(\vartheta_m,\vartheta_{m+2})+\kappa(\vartheta_{m+2},\vartheta_{m+3})+\cdots+\kappa(\vartheta_{m+2n-1},\vartheta_{m+2n})\\ &\preceq 3\gamma^m\kappa(\vartheta_0,\vartheta_1)+\{\gamma^{m+2}+\gamma^{m+3}+\cdots+\gamma^{m+2n-1}\}\kappa(\vartheta_0,\vartheta_1) \quad \text{(by using (12) and (16))}\\ &\preceq \left\{\left[3\gamma^m+\frac{\gamma^{m+2}}{1-\gamma}\right]\kappa(\vartheta_0,\vartheta_1)\right\}\downarrow 0.\end{aligned}$$

Hence $\{\vartheta_m\}$ is Q-Cauchy in W, so then there exists a_m in Q such that $a_m\downarrow 0$ and

$$\kappa(\vartheta_m,\vartheta_{m+p})\preceq a_m. \tag{17}$$

for all m and p. Let $F_3(W)$ be Q-complete, so ∃ some $s\in F_3(W)$ s.t.

$$F_3\varsigma_{2m}=\vartheta_{2m}\xrightarrow{\kappa,Q} s, \tag{18}$$

then by using Lemma 1.12., we get $\vartheta_m\xrightarrow{\kappa,Q} s$ i.e. ∃ a sequence $\{b_m\}\in Q$ s.t. $b_m\downarrow 0$ and

$$\kappa(\vartheta_m,s)\preceq b_m. \tag{19}$$

Further since $s\in F_3(W)$ there exists $g\in W$ s.t. $F_3g=s$. Now

$$\kappa(F_1g,\vartheta_{2m})=\kappa(F_1g,F_2\varsigma_{2m-1})\preceq\gamma H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4),$$

where

$$\begin{aligned}H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4)&\in\{\kappa(F_3g,F_4\varsigma_{2m-1}),\kappa(F_1g,F_3g),\kappa(F_2\varsigma_{2m-1},F_4\varsigma_{2m-1}),\kappa(F_1g,F_4\varsigma_{2m-1})\}\\ &=\{\kappa(s,\vartheta_{2m-1}),\kappa(F_1g,s),\kappa(\vartheta_{2m},\vartheta_{2m-1}),\kappa(F_1g,\vartheta_{2m-1})\}\end{aligned}$$

Now we have the following four possibilities:

(i)If $H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4)=\kappa(s,\vartheta_{2m-1})$, then

$$\begin{aligned}\kappa(F_1g,\vartheta_{2m})&\preceq\gamma\kappa(s,\vartheta_{2m-1})\preceq\gamma b_{2m-1} \quad \text{(by using (19))}\\ &\preceq\gamma b_m\downarrow 0 \quad (\because\ b_{2m-1}\preceq b_m).\end{aligned}$$

(ii) If $H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4)=\kappa(F_1g,s)$, then

$$\begin{aligned}\kappa(F_1g,\vartheta_{2m})&\preceq\gamma\kappa(F_1g,s)\\ &\preceq\gamma[\kappa(F_1g,\vartheta_{2m})+\kappa(\vartheta_{2m},\vartheta_{2m+1})+\kappa(\vartheta_{2m+1},s)] \quad \text{(by rectangular inequality)}\\ (1-\gamma)\kappa(F_1g,\vartheta_{2m})&\preceq\gamma a_{2m}+\gamma b_{2m+1} \quad \text{(by using (17) and (19))}\\ &\preceq\left\{\frac{\gamma}{1-\gamma}(a_m+b_m)\right\}\downarrow 0 \quad (\because a_{2m}\preceq a_m \text{ and } b_{2m+1}\preceq b_m).\end{aligned}$$

(iii) If $H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4)=\kappa(\vartheta_{2m},\vartheta_{2m-1})$, then

$$\begin{aligned}\kappa(F_1g,\vartheta_{2m})&\preceq\gamma\kappa(\vartheta_{2m},\vartheta_{2m-1})\preceq\gamma a_{2m-1} \quad \text{(by using(17))}\\ &\preceq\gamma a_m\downarrow 0.\end{aligned}$$

(iv) If $H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4)=\kappa(F_1g,\vartheta_{2m-1})$, then by rectangular inequality, we have

$$\begin{aligned}\kappa(F_1g,\vartheta_{2m})&\preceq\gamma\kappa(F_1g,\vartheta_{2m-1})\\ &\preceq\gamma[\kappa(F_1g,\vartheta_{2m})+\kappa(\vartheta_{2m},\vartheta_{2m+1})+\kappa(\vartheta_{2m+1},\vartheta_{2m-1})]\\ (1-\gamma)\kappa(F_1g,\vartheta_{2m})&\preceq\gamma(a_{2m}+a_{2m-1}) \quad \text{(using (17))}\\ \kappa(F_1g,\vartheta_{2m})&\preceq\left\{\frac{2\gamma}{1-\gamma}a_m\right\}\downarrow 0 \quad (\because a_{2m}\preceq a_{2m-1}\preceq a_m).\end{aligned}$$

We get $\vartheta_{2m}\xrightarrow{\kappa,Q}F_1g$. Since by Lemma 1.13. and from (18), we deduce that $F_1g=s$. Thus

$$F_1g=F_3g=s. \tag{20}$$

Since $s\in F_1(W)\subseteq F_4(W)$, so ∃ $r\in W$ s.t. $F_4r=s$. Now, we shall show that $F_2r=s$. Now by rectangular inequality, we have

$$\begin{aligned}\kappa(s,F_2r)&\preceq\kappa(s,\vartheta_{2m})+\kappa(\vartheta_{2m},\vartheta_{2m+1})+\kappa(\vartheta_{2m+1},F_2r)\\ &\preceq\kappa(s,\vartheta_{2m})+a_{2m}+\kappa(F_1\varsigma_{2m},F_2r) \quad \text{(by using (8) and(17))}\\ &\preceq b_{2m}+a_{2m}+\gamma H_{\varsigma_{2m},r}(F_1,F_2,F_3,F_4) \quad \text{(by using (7) and (19))}\\ &\preceq b_m+a_m+\gamma H_{\varsigma_{2m},r}(F_1,F_2,F_3,F_4) \quad (\because b_{2m}\preceq b_m \text{ and } a_{2m}\preceq a_m),\end{aligned}$$

where

$$H_{\varsigma_{2m},r}(F_1,F_2,F_3,F_4)\in\{\kappa(F_3\varsigma_{2m},F_4r),\kappa(F_1\varsigma_{2m},F_3\varsigma_{2m}),\kappa(F_2r,F_4r),\kappa(F_1\varsigma_{2m},F_4r)\}$$

$$= \{\kappa(F_3\varsigma_{2m}, s), \kappa(F_1\varsigma_{2m}, F_3\varsigma_{2m}), \kappa(F_2 r, s), \kappa(F_1\varsigma_{2m}, s)\}$$
$$= \{\kappa(\vartheta_{2m}, s), \kappa(\vartheta_{2m+1}, \vartheta_{2m}), \kappa(F_2 r, s), \kappa(\vartheta_{2m+1}, s)\}$$

There are four possibilities:

Case 1. If $H_{\varsigma_{2m},r}(F_1, F_2, F_3, F_4) = \kappa(\vartheta_{2m}, s)$, then we have

$$\kappa(s,\ F_2 r) \preceq b_m + a_m + \gamma\kappa(\vartheta_{2m}, s)$$
$$\preceq b_m + a_m + \gamma b_{2m} \text{ (by using (19))}$$
$$= (1\ +\ \gamma)b_m + a_m\ (\because b_{2m} \preceq b_m).$$

Case 2. If $H_{\varsigma_{2m},r}(F_1, F_2, F_3, F_4) = \kappa(\vartheta_{2m+1}, \vartheta_{2m})$, then we have

$$\kappa(s,\ F_2 r) \preceq b_m + a_m + \gamma\kappa(\vartheta_{2m+1}, \vartheta_{2m})$$
$$\preceq b_m + a_m + \gamma b a_{2m} \text{ (by using (17))}$$
$$= b_m + (1\ +\ \gamma)\ a_m\ \ (\because a_{2m} \preceq a_m).$$

Case 3. If $H_{\varsigma_{2m},r}(F_1, F_2, F_3, F_4) = \kappa(F_2 r, s)$, then we have

$$\kappa(s,\ F_2 r) \preceq b_m + a_m + \gamma\kappa(F_2 r, s)$$
$$(1-\gamma)\kappa(s,\ F_2 r) \preceq b_m + a_m$$
$$\kappa(s,\ F_2 r) \preceq \frac{1}{1-\gamma}(b_m + a_m).$$

Case 4. If $H_{\varsigma_{2m},r}(F_1, F_2, F_3, F_4) = \kappa(\vartheta_{2m+1}, s)$, then we have

$$\kappa(s,\ F_2 r) \preceq b_m + a_m + \gamma\kappa(\vartheta_{2m+1}, s)$$
$$\preceq b_m + a_m + \gamma b_{2m+1} \text{ (by using (19))}$$
$$\preceq (1\ +\ \gamma)b_m + a_m\ (\because b_{2m+1} \preceq b_m).$$

Implies $\kappa(s,\ F_2 r) = 0$, i.e. $F_2 r = s$. Thus $F_2 r = F_4 r = s$. Consequently $\{F_1, F_3\}$ and $\{F_2, F_4\}$ have a common point of coincidence from W. Since the pairs $\{F_1, F_3\}$ and $\{F_2, F_4\}$ are WC, it follows that

$$F_1 s = F_1 F_3 g = F_3 F_1 g = F_3 s = m_1 \text{ (say)}, \qquad (21)$$

and also

$$F_2 s = F_2 F_4 r = F_4 F_2 r = F_4 s = m_2 \text{ (say)}. \qquad (22)$$

Now

$$\kappa(m_1, m_2) = \kappa(F_1 s, F_2 s) \preceq \gamma H_{s,s}(F_1, F_2, F_3, F_4),$$

where

$$H_{s,s}(F_1, F_2, F_3, F_4) \in \{\kappa(F_3 s, F_4 s), \kappa(F_1 s, F_3 s), \kappa(F_2 s, F_4 s), \kappa(F_1 s, F_4 s)\}$$
$$= \{0, \kappa(m_1, m_2)\} \text{ (by using (21) and (22)).}$$

Thus $\kappa(m_1, m_2) \preceq \gamma\kappa(m_1, m_2)$, we get $\kappa(m_1, m_2) = 0$. Hence $m_1 = m_2$. $\therefore F_1 s = F_2 s = F_3 s = F_4 s$. Now we have to prove that $F_2 s = s$. We have

$$\kappa(s, F_2 s) = \kappa(F_1 g, F_2 s) \preceq \gamma H_{g,s}(F_1, F_2, F_3, F_4),$$

where

$$H_{g,s}(F_1, F_2, F_3, F_4) \in \{\kappa(F_3 g, F_4 s), \kappa(F_1 g, F_3 g), \kappa(F_2 s, F_4 s), \kappa(F_1 g, F_4 s)\}$$
$$= \{\kappa(s, F_4 s), \kappa(s, s), \kappa(F_2 s, F_4 s), \kappa(s, F_4 s)\} \text{ (by using (20),(21) and (22))}$$
$$= \{0, \kappa(s, F_2 s)\} \text{ (by using (21).}$$

That is $\kappa(s, F_2 s) \preceq \gamma\kappa(s, F_2 s)$, which implies that $\kappa(s, F_2 s) = 0$. Hence $s = F_2 s = F_4 s$ i.e. $F_1 s = F_2 s = F_3 s = F_4 s = s$. Thus s is a c.f.p. of F_1, F_2, F_3 & F_4. Next we shall show the uniqueness of s. Assume some other c.f.p. s^* of F_1, F_2, F_3 & F_4 i.e $F_1 s^* = F_2 s^* = F_3 s^* = F_4 s^* = s^*$. Then

$$\kappa(s, s^*) = \kappa(F_1 s, F_2 s\ *) \preceq \gamma H_{s,s^*}(F_1, F_2, F_3, F_4).$$

where

$$H_{s,s^*}(F_1, F_2, F_3, F_4) \in \{\kappa(F_3 s, F_4 s^*), \kappa(F_1 s, F_3 s), \kappa(F_2 s^*, F_4 s^*), \kappa(F_1 s, F_4 s^*)\}$$
$$= \{0, \kappa(s, s^*)\}.$$

Implies $\kappa(s, s^*) = 0$ i.e. $s = s^*$. Hence s is a c.f.p of F_1, F_2, F_3 & F_4 which is unique. The proofs for the cases in which $F_1(W), F_2(W)$ and $F_4(W)$ is complete are similar.

Corollary 2.2. Let (W, κ, Q) be a vector valued r.m.s with Q-Archimedean. Suppose the F_1, F_2, F_3 & F_4 are self-mappings on W with $F_1(W) \subseteq F_4(W)$ and $F_2(W) \subseteq F_3(W)$. Let the following condition hold:

$$\kappa(F_1\varsigma, F_2\vartheta) \preceq \gamma\kappa(F_3\varsigma, F_4\vartheta)$$
$$\kappa(F_3\varsigma, F_2\vartheta) \preceq \gamma\kappa(F_3\varsigma, F_4\vartheta)$$

for all $\varsigma, \vartheta \in W$ and $\gamma \in [0, 1)$. Let the pairs $\{F_1, F_3\}$ and $\{F_2, F_4\}$ are given to be WC and if one of $F_1(W), F_2(W), F_3(W)$ or $F_4(W)$ is Q-complete subspace of W then F_1, F_2, F_3 & F_4 have c.f.p which is unique.

Proof. It follows directly from Theorem 2.1.

Corollary 2.3. Let (W, κ, Q) be a vector valued r.m.s with Q-Archimedean. Suppose that F_1, F_2 and F_3 are self-mappings on W, with $F_1(W) \cup F_2(W) \subseteq F_3(W)$. Let the following condition hold:

$$\kappa(F_1\varsigma, F_2\vartheta) \preceq \gamma H_{\varsigma,\vartheta}(F_1, F_2, F_3)$$
$$\kappa(F_3\varsigma, F_2\vartheta) \preceq \gamma H_{\varsigma,\vartheta}(F_1, F_2, F_3)$$

for all $\varsigma, \vartheta \in W$ and $\gamma \in [0, 1)$, where

$$H_{\varsigma,\vartheta}(F_1, F_2, F_3) \in \{\kappa(F_3\varsigma, F_3\vartheta), \kappa(F_1\varsigma, F_3\varsigma), \kappa(F_2\vartheta, F_3\vartheta), \kappa(F_1\varsigma, F_3\vartheta)\}.$$

Let the pairs $\{F_1, F_3\}$ and $\{F_2, F_3\}$ are given to be WC and if one of $F_1(W), F_2(W)$ or $F_3(W)$ is Q-complete subspace of W then F_1, F_2 and F_3 have c.f.p which is unique.

Proof. Result follows by taking mapping $F_3 = F_4$ in Theorem 2.1.

Theorem 2.4. Let (W, κ, Q) be a vector valued r.m.s with Q-Archimedean. Suppose the self-maps F_1, F_2, F_3 & F_4 on W satisfy the following conditions:

(i) for all $\varsigma, \vartheta \in W$, and $\gamma \in [0, 1)$

$$\left.\begin{aligned}\kappa(F_1\varsigma, F_2\vartheta) \preceq \gamma H_{\varsigma,\vartheta}(F_1, F_2, F_3, F_4)\\ \kappa(F_3\varsigma, F_2\vartheta) \preceq \gamma H_{\varsigma,\vartheta}(F_1, F_2, F_3, F_4)\end{aligned}\right\}$$

where $H_{\varsigma,\vartheta}(F_1, F_2, F_3, F_4) \in \Big\{\frac{1}{2}[\kappa(F_3\varsigma, F_4\vartheta) + \kappa(F_1\varsigma, F_3\varsigma)], \frac{1}{2}[\kappa(F_3\varsigma, F_4\vartheta) + \kappa(F_2\vartheta, F_4\vartheta)],$
$\frac{1}{2}[\kappa(F_3\varsigma, F_4\vartheta) + \kappa(F_1\varsigma, F_4\vartheta)], \frac{1}{2}[\kappa(F_2\vartheta, F_4\vartheta), \kappa(F_1\varsigma, F_4\vartheta)]\Big\}.$

(ii) $F_1(W) \subseteq F_4(W)$ and $F_2(W) \subseteq F_3(W)$.

(iii) $F_1(W), F_2(W), F_3(W)$ or $F_4(W)$ is Q-complete subspace of W.

Then the pairs $\{F_1, F_3\}$ and $\{F_2, F_4\}$ have unique point of coincidence. Moreover, if the pairs $\{F_1, F_3\}$ and $\{F_2, F_4\}$ are WC then F_1, F_2, F_3 & F_4 have c.f.p which is unique.

Proof. We define the sequences $\{\varsigma_m\}$ and $\{\vartheta_m\}$ follows, as in the proof of the Theorem 2.1. Firstly, we show that

$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) \preceq \lambda\kappa(\vartheta_{2m}, \vartheta_{2m+1}), \text{ where } \lambda \in [0,1).$$

We have

$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) = \kappa(F_1\varsigma_{2m}, F_2\varsigma_{2m+1}) \preceq \gamma H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4),$$

where

$$\begin{aligned}H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) \in &\Big\{\frac{1}{2}[\kappa(F_3\varsigma_{2m}, F_4\varsigma_{2m+1}) + \kappa(F_1\varsigma_{2m}, F_3\varsigma_{2m})], \frac{1}{2}[\kappa(F_3\varsigma_{2m}, F_4\varsigma_{2m+1})\\ &+\kappa(F_2\varsigma_{2m+1}, F_4\varsigma_{2m+1})], \frac{1}{2}[\kappa(F_3\varsigma_{2m}, F_4\varsigma_{2m+1}) + \kappa(F_1\varsigma_{2m}, F_4\varsigma_{2m+1})],\\ &\frac{1}{2}[\kappa(F_2\varsigma_{2m+1}, F_4\varsigma_{2m+1}), \kappa(F_1\varsigma_{2m}, F_4\varsigma_{2m+1})]\Big\}\\ &= \Big\{\frac{1}{2}[\kappa(\vartheta_{2m}, \vartheta_{2m+1}) + \kappa(\vartheta_{2m+1}, \vartheta_{2m})], \frac{1}{2}[\kappa(\vartheta_{2m}, \vartheta_{2m+1})\\ &+\kappa(\vartheta_{2m+2}, \vartheta_{2m+1})], \frac{1}{2}[\kappa(\vartheta_{2m}, \vartheta_{2m+1}) + \kappa(\vartheta_{2m+1}, \vartheta_{2m+1})],\\ &\frac{1}{2}[\kappa(\vartheta_{2m+2}, \vartheta_{2m+1}) + \kappa(\vartheta_{2m+1}, \vartheta_{2m+1})]\Big\}\\ &= \{\kappa(\vartheta_{2m}, \vartheta_{2m+1}), \frac{1}{2}[\kappa(\vartheta_{2m}, \vartheta_{2m+1}) + \kappa(\vartheta_{2m+2}, \vartheta_{2m+1})],\\ &\frac{1}{2}\kappa(\vartheta_{2m}, \vartheta_{2m+1}), \frac{1}{2}\kappa(\vartheta_{2m+2}, \vartheta_{2m+1})\Big\}. \qquad (24)\end{aligned}$$

The possible four cases are:

(i) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) = \kappa(\vartheta_{2m}, \vartheta_{2m+1})$. Then we have

$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) \preceq \gamma\kappa(\vartheta_{2m}, \vartheta_{2m+1}).$$

(ii) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) = \frac{1}{2}[\kappa(\vartheta_{2m}, \vartheta_{2m+1}) + \kappa(\vartheta_{2m+2}, \vartheta_{2m+1})]$. Then

$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) \preceq \frac{\gamma}{2}[\kappa(\vartheta_{2m}, \vartheta_{2m+1}) + \kappa(\vartheta_{2m+2}, \vartheta_{2m+1})]$$
$$\left(1 - \frac{\gamma}{2}\right)\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) \preceq \frac{\gamma}{2}\kappa(\vartheta_{2m}, \vartheta_{2m+1})$$
$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) \preceq \frac{\gamma}{2-\gamma}\kappa(\vartheta_{2m}, \vartheta_{2m+1})$$

where $\frac{\gamma}{2-\gamma} < 1$.

(iii) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1, F_2, F_3, F_4) = \frac{1}{2}\kappa(\vartheta_{2m}, \vartheta_{2m+1})$, then

$$\kappa(\vartheta_{2m+1}, \vartheta_{2m+2}) \preceq \frac{\gamma}{2}\kappa(\vartheta_{2m}, \vartheta_{2m+1}).$$

(iv) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1,F_2,F_3,F_4)=\frac{1}{2}\kappa(\vartheta_{2m+2},\vartheta_{2m+1})$ then

$$\kappa(\vartheta_{2m+1},\vartheta_{2m+2})\leq\frac{\gamma}{2}\kappa(\vartheta_{2m+2},\vartheta_{2m+1}),$$

implies $\kappa(\vartheta_{2m+1},\vartheta_{2m+2})=0$. Thus, we get

$$\kappa(\vartheta_{2m+1},\vartheta_{2m+2})\leq\lambda\kappa(\vartheta_{2m},\vartheta_{2m+1}) \tag{25}$$

where $0\leq\lambda\in\left\{\gamma,\frac{\gamma}{2},\frac{\gamma}{2-\gamma}\right\}<1$.

Similarly, we can prove

$$\kappa(\vartheta_{2m+2},\vartheta_{2m+3})\leq\lambda\kappa(\vartheta_{2m+1},\vartheta_{2m+2}). \tag{26}$$

Thus, for all $m\geq 1$ and from (25) and (26),

$$\kappa(\vartheta_m,\vartheta_{m+1})\leq\lambda\kappa(\vartheta_{m-1},\vartheta_m). \tag{27}$$

Then from (27), we obtain

$$\kappa(\vartheta_m,\vartheta_{m+1})\leq\lambda^m\kappa(\vartheta_0,\vartheta_1).$$

Since $\gamma\geq\lambda$, we get

$$\kappa(\vartheta_m,\vartheta_{m+1})\leq\gamma^m\kappa(\vartheta_0,\vartheta_1). \tag{28}$$

Now by using (8), (23), we obtain

$$\kappa(\vartheta_{2m},\vartheta_{2m+2})\leq\kappa(F_3\varsigma_{2m},F_2\varsigma_{2m+1})\leq\gamma H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1,F_2,F_3,F_4),$$

where

$$H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1,F_2,F_3,F_4)\in\Big\{\kappa(\vartheta_{2m},\vartheta_{2m+1}),\frac{1}{2}[\kappa(\vartheta_{2m},\vartheta_{2m+1})+\kappa(\vartheta_{2m+2},\vartheta_{2m+1})],\\ \frac{1}{2}\kappa(\vartheta_{2m},\vartheta_{2m+1}),\frac{1}{2}\kappa(\vartheta_{2m+2},\vartheta_{2m+1})\Big\}.$$

The possible four cases are:

(i) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1,F_2,F_3,F_4)=\kappa(\vartheta_{2m},\vartheta_{2m+1})$. Then

$$\begin{aligned}\kappa(\vartheta_{2m},\vartheta_{2m+2})&\leq\gamma\kappa(\vartheta_{2m},\vartheta_{2m+1})\\&\leq\gamma^{2m+1}\kappa(\vartheta_0,\vartheta_1)\ \text{(by using (28))}.\end{aligned}$$

(ii) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1,F_2,F_3,F_4)=\frac{1}{2}[\kappa(\vartheta_{2m},\vartheta_{2m+1})+\kappa(\vartheta_{2m+2},\vartheta_{2m+1})]$. Then

$$\begin{aligned}\kappa(\vartheta_{2m},\vartheta_{2m+2})&\leq\frac{\gamma}{2}[\kappa(\vartheta_{2m},\vartheta_{2m+1})+\kappa(\vartheta_{2m+2},\vartheta_{2m+1})]\\&\leq\frac{\gamma}{2}[(\gamma^{2m}+\gamma^{2m+1})\,\kappa(\vartheta_0,\vartheta_1)]\\&\leq\frac{\gamma}{2}[2\gamma^{2m}\kappa(\vartheta_0,\vartheta_1)]\quad(\because\gamma^{2m}\geq\gamma^{2m+1})\\&=\gamma^{2m+1}\kappa(\vartheta_0,\vartheta_1).\end{aligned}$$

(iii) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1,F_2,F_3,F_4)=\frac{1}{2}\kappa(\vartheta_{2m},\vartheta_{2m+1})$, then

$$\begin{aligned}\kappa(\vartheta_{2m},\vartheta_{2m+2})&\leq\frac{\gamma}{2}\kappa(\vartheta_{2m},\vartheta_{2m+1})\\&\leq\gamma^{2m+1}\kappa(\vartheta_0,\vartheta_1).\end{aligned}$$

(iv) If $H_{\varsigma_{2m},\varsigma_{2m+1}}(F_1,F_2,F_3,F_4)=\frac{1}{2}\kappa(\vartheta_{2m+2},\vartheta_{2m+1})$, then

$$\begin{aligned}\kappa(\vartheta_{2m},\vartheta_{2m+2})&\leq\frac{\gamma}{2}\kappa(\vartheta_{2m+2},\vartheta_{2m+1})\\&\leq\gamma\gamma^{2m+1}\kappa(\vartheta_0,\vartheta_1)\quad(\because\ \tfrac{\gamma}{2}\leq\gamma)\\&\leq\gamma^{2m+2}\kappa(\vartheta_0,\vartheta_1)\\&\leq\gamma^{2m+1}\kappa(\vartheta_0,\vartheta_1)\quad(\because\gamma^{2m+1}\geq\gamma^{2m+2}).\end{aligned}$$

We get

$$\kappa(\vartheta_{2m},\vartheta_{2m+2})\leq\gamma^{2m+1}\kappa(\vartheta_0,\vartheta_1) \tag{29}$$

Again by rectangular inequality, we have

$$\begin{aligned}\kappa(\vartheta_{2m+1},\vartheta_{2m+3})&\leq\kappa(\vartheta_{2m+1},\vartheta_{2m})+\kappa(\vartheta_{2m},\vartheta_{2m+2})+\kappa(\vartheta_{2m+2},\vartheta_{2m+3})\\&\leq\gamma^{2m}\kappa(\vartheta_0,\vartheta_1)+\gamma^{2m+1}\kappa(\vartheta_0,\vartheta_1)+\gamma^{2m+2}\kappa(\vartheta_0,\vartheta_1)\\&\leq 3\gamma^{2m}\kappa(\vartheta_0,\vartheta_1)\quad(\because\ 0\leq\gamma<1).\end{aligned} \tag{30}$$

Also from (29), we can write

$$\kappa(\vartheta_{2m},\vartheta_{2m+2})\leq\gamma^{2m+1}\kappa(\vartheta_0,\vartheta_1)\leq 3\gamma^{2m}\kappa(\vartheta_0,\vartheta_1). \tag{31}$$

Therefore from (30) and (31), we obtain

$$\kappa(\vartheta_m,\vartheta_{m+2})\leq 3\gamma^m\kappa(\vartheta_0,\vartheta_1), \tag{32}$$

for all m ≥ 1. For $\{\vartheta_m\}$, we consider $\kappa(\vartheta_m,\vartheta_{m+p})$ in two cases.

Case 1. If p is odd say $2n+1$, $n\in N\cup\{0\}$, then by rectangular inequality

$$\kappa(\vartheta_m,\vartheta_{m+p})\leq\kappa(\vartheta_m,\vartheta_{m+1})+\kappa(\vartheta_{m+1},\vartheta_{m+2})+\kappa(\vartheta_{m+2},\vartheta_{m+p})$$

$$\leq \kappa(\vartheta_m,\vartheta_{m+1})+\kappa(\vartheta_{m+1},\vartheta_{m+2})+\kappa(\vartheta_{m+2},\vartheta_{m+3})+\kappa(\vartheta_{m+3},\vartheta_{m+4})+\kappa(\vartheta_{m+4},\vartheta_{m+p})$$
$$\leq \kappa(\vartheta_m,\vartheta_{m+1})+\kappa(\vartheta_{m+1},\vartheta_{m+2})+\cdots+\kappa(\vartheta_{m+2n-1},\vartheta_{m+2n})+\kappa(\vartheta_{m+2n},\vartheta_{m+p})$$
$$\leq \gamma^m\kappa(\vartheta_0,\vartheta_1)+\gamma^{2m+1}\kappa(\vartheta_0,\vartheta_1)+\cdots+\gamma^{2m+2}\kappa(\vartheta_0,\vartheta_1) \quad \text{(by using (28))}$$
$$\leq \left\{\frac{\gamma^m}{1-\gamma}\kappa(\vartheta_0,\vartheta_1)\right\}\downarrow 0.$$

Case 2. If p is even say $2n$, $n\in N$, then by rectangular inequality

$$\kappa(\vartheta_m,\vartheta_{m+p})\leq \kappa(\vartheta_m,\vartheta_{m+2})+\kappa(\vartheta_{m+2},\vartheta_{m+3})+\kappa(\vartheta_{m+3},\vartheta_{m+p})$$
$$\leq \kappa(\vartheta_m,\vartheta_{m+2})+\kappa(\vartheta_{m+2},\vartheta_{m+3})+\kappa(\vartheta_{m+3},\vartheta_{m+4})+\kappa(\vartheta_{m+4},\vartheta_{m+5})+\kappa(\vartheta_{m+5},\vartheta_{m+p})$$
$$\leq \kappa(\vartheta_m,\vartheta_{m+2})+\kappa(\vartheta_{m+2},\vartheta_{m+3})+\cdots+\kappa(\vartheta_{m+2n-1},\vartheta_{m+2n})$$
$$\leq 3\gamma^m\kappa(\vartheta_0,\vartheta_1)+\{\gamma^{m+2}+\gamma^{m+3}+\cdots+\gamma^{m+2n-1}\}\kappa(\vartheta_0,\vartheta_1) \quad \text{(by using (28) and (32))}$$
$$\leq \left\{\left[3\gamma^m+\frac{\gamma^{m+2}}{1-\gamma}\right]\kappa(\vartheta_0,\vartheta_1)\right\}\downarrow 0.$$

Hence $\{\vartheta_m\}$ is Q-Cauchy in W, so then there exists a_m in Q such that $a_m\downarrow 0$ and

$$\kappa(\vartheta_m,\vartheta_{m+p})\leq a_m. \tag{33}$$

for all m and p. Let $F_3(W)$ be Q-complete, so ∃ some $s\in F_3(W)$ s.t.

$$F_3\varsigma_{2m}=\vartheta_{2m}\xrightarrow{\kappa,Q}s, \tag{34}$$

then by using Lemma 1.12., we get $\vartheta_m\xrightarrow{\kappa,Q}s$ i.e. ∃ a sequence $\{b_m\}\in Q$ s.t. $b_m\downarrow 0$, and

$$\kappa(\vartheta_m,s)\leq b_m. \tag{35}$$

Further since $s\in F_3(W)$ there exists $g\in W$ s.t. $F_3g=s$. Now

$$\kappa(F_1g,\vartheta_{2m})=\kappa(F_1g,F_2\varsigma_{2m-1})\leq\gamma H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4),$$

where

$$H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4)\in\Big\{\frac{1}{2}[\kappa(F_3g,F_4\varsigma_{2m-1})+\kappa(F_1g,F_3g)],\frac{1}{2}[\kappa(F_3g,F_4\varsigma_{2m-1})$$
$$+\kappa(F_2\varsigma_{2m-1},F_4\varsigma_{2m-1})],\frac{1}{2}[\kappa(F_3g,F_4\varsigma_{2m-1})+\kappa(F_1g,F_4\varsigma_{2m-1})],$$
$$\frac{1}{2}[\kappa(F_2\varsigma_{2m-1},F_4\varsigma_{2m-1}),\kappa(F_1g,F_4\varsigma_{2m-1})]\Big\}$$
$$=\Big\{\frac{1}{2}[\kappa(s,\vartheta_{2m-1})+\kappa(F_1g,s)],\frac{1}{2}[\kappa(s,\vartheta_{2m-1})$$
$$+\kappa(\vartheta_{2m},\vartheta_{2m-1})],\frac{1}{2}[\kappa(s,\vartheta_{2m-1})+\kappa(F_1g,\vartheta_{2m-1})],$$
$$\frac{1}{2}[\kappa(\vartheta_{2m},\vartheta_{2m-1})+\kappa(F_1g,\vartheta_{2m-1})]\Big\}.$$

Now we have the following four possibilities:

(i) If $H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4)=\frac{1}{2}[\kappa(s,\vartheta_{2m-1})+\kappa(F_1g,s)]$, then

$$\kappa(F_1g,\vartheta_{2m})\leq\frac{\gamma}{2}[\kappa(s,\vartheta_{2m-1})+\kappa(F_1g,s)]$$
$$\leq\frac{\gamma}{2}[b_{2m-1}+\kappa(F_1g,\vartheta_{2m})+\kappa(\vartheta_{2m},\vartheta_{2m+1})+\kappa(\vartheta_{2m+1},s)]$$
$$\frac{2-\gamma}{2}\kappa(F_1g,\vartheta_{2m})\leq\frac{\gamma}{2}[b_{2m-1}+a_{2m}+b_{2m+1}] \quad \text{(by using(33) and (35))}$$

i.e. $\kappa(F_1g,\vartheta_{2m})\leq\left\{\frac{\gamma}{2-\gamma}[a_m+b_m]\right\}\downarrow 0$ ($\because a_{2m}\leq a_m$ and $\{b_{2m-1},b_{2m+1}\}\leq b_m$).

(ii) If $H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4)=\frac{1}{2}[\kappa(s,\vartheta_{2m-1})+\kappa(\vartheta_{2m},\vartheta_{2m-1})]$, then

$$\kappa(F_1g,\vartheta_{2m})\leq\frac{\gamma}{2}[\kappa(s,\vartheta_{2m-1})+\kappa(\vartheta_{2m},\vartheta_{2m-1})]$$
$$\leq\frac{\gamma}{2}[b_{2m-1}+a_{2m-1}]$$
$$\leq\left\{\frac{\gamma}{2}[b_m+a_m]\right\}\downarrow 0.$$

(iii) If $H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4)=\frac{1}{2}[\kappa(s,\vartheta_{2m-1})+\kappa(F_1g,\vartheta_{2m-1})]$, then

$$\kappa(F_1g,\vartheta_{2m})\leq\frac{\gamma}{2}[\kappa(s,\vartheta_{2m-1})+\kappa(F_1g,\vartheta_{2m-1})]$$
$$\leq\frac{\gamma}{2}[b_{2m-1}+\kappa(F_1g,\vartheta_{2m})+\kappa(\vartheta_{2m},\vartheta_{2m+1})+\kappa(\vartheta_{2m+1},\vartheta_{2m-1})]$$
$$\frac{2-\gamma}{2}\kappa(F_1g,\vartheta_{2m})\leq\frac{\gamma}{2}[b_{2m-1}+a_{2m}+b_{2m-1}]$$

i.e. $\kappa(F_1g,\vartheta_{2m})\leq\left\{\frac{\gamma}{2-\gamma}(2a_m+b_m)\right\}\downarrow 0.$

(iv) If $H_{g,\varsigma_{2m-1}}(F_1,F_2,F_3,F_4)=\frac{1}{2}[\kappa(\vartheta_{2m},\vartheta_{2m-1})+\kappa(F_1g,\vartheta_{2m-1})]$, then

$$\kappa(F_1g,\vartheta_{2m})\leq\frac{\gamma}{2}[\kappa(\vartheta_{2m},\vartheta_{2m-1})+\kappa(F_1g,\vartheta_{2m-1})]$$
$$\leq\frac{\gamma}{2}[a_{2m-1}+\kappa(F_1g,\vartheta_{2m})+\kappa(\vartheta_{2m},\vartheta_{2m+1})+\kappa(\vartheta_{2m+1},\vartheta_{2m-1})]$$
$$\frac{2-\gamma}{2}\kappa(F_1g,\vartheta_{2m})\leq\frac{\gamma}{2}[a_{2m-1}+a_{2m}+a_{2m-1}]$$

i.e. $\kappa(F_1g,\vartheta_{2m})\leq\left\{\frac{3\gamma}{2-\gamma}a_m\right\}\downarrow 0$.

So we can conclude that $\vartheta_{2m}\xrightarrow{\kappa,Q}F_1g$. By Lemma 1.13. and from (34), we deduce that $F_1g=s$. Thus

$$F_1g=F_3g=s. \tag{36}$$

Since $s\in F_1(W)\subseteq F_4(W)$, so $\exists\ r\in W$ s.t.

$$F_4r=s. \tag{37}$$

Now, we shall show that $F_2r=s$. Now by rectangular inequality, we have

$$\kappa(s,\ F_2r)\leq\kappa(s,\vartheta_{2m})+\kappa(\vartheta_{2m},\vartheta_{2m+1})+\kappa(\vartheta_{2m+1},\ F_2r)$$
$$\leq\kappa(s,F_3\varsigma_{2m})+a_{2m}+\kappa(F_1\varsigma_{2m},\ F_2r)\text{ (by using (8) and(33))}$$
$$\leq b_{2m}+a_{2m}+\gamma H_{\varsigma_{2m},r}(F_1,F_2,F_3,F_4)\text{ (by using (7) and (19))}$$
$$\leq b_m+a_m+\gamma H_{\varsigma_{2m},r}(F_1,F_2,F_3,F_4)\ (\because b_{2m}\leq b_m\text{ and }a_{2m}\leq a_m),$$

where

$$H_{\varsigma_{2m},r}(F_1,F_2,F_3,F_4)\in\Big\{\frac{1}{2}[\kappa(F_3\varsigma_{2m},F_4r)+\kappa(F_1\varsigma_{2m},F_3\varsigma_{2m})],\frac{1}{2}[\kappa(F_3\varsigma_{2m},F_4r)+\kappa(F_2r,F_4r)],\frac{1}{2}[\kappa(F_3\varsigma_{2m},F_4r)+\kappa(F_1\varsigma_{2m},F_4r)],\frac{1}{2}[\kappa(F_2r,F_4r),\kappa(F_1\varsigma_{2m},F_4r)]\Big\}$$
$$=\Big\{\frac{1}{2}[\kappa(\vartheta_{2m},s)+\kappa(\vartheta_{2m+1},\vartheta_{2m})],\frac{1}{2}[\kappa(\vartheta_{2m},s)+\kappa(F_2r,s)],\frac{1}{2}[\kappa(\vartheta_{2m},s)+\kappa(\vartheta_{2m+1},s)],\frac{1}{2}[\kappa(F_2r,s)+\kappa(\vartheta_{2m+1},s)]\Big\}\text{ (using (8) and (37)).}$$

The possible four cases are:

(i) If $H_{\varsigma_{2m},r}(F_1,F_2,F_3,F_4)=\frac{1}{2}[\kappa(\vartheta_{2m},s)+\kappa(\vartheta_{2m+1},\vartheta_{2m})]$. Then we have

$$\kappa(s,\ F_2r)\leq b_m+a_m+\frac{\gamma}{2}[\kappa(\vartheta_{2m},s)+\kappa(\vartheta_{2m+1},\vartheta_{2m})$$
$$\leq b_m+a_m+\frac{\gamma}{2}[b_{2m}+a_{2m}]$$
$$\leq\frac{2+\gamma}{2}[a_m+b_m].$$

(ii) If $H_{\varsigma_{2m},r}(F_1,F_2,F_3,F_4)=\frac{1}{2}[\kappa(\vartheta_{2m},s)+\kappa(F_2r,s)]$. Then we have

$$\kappa(s,\ F_2r)\leq b_m+a_m+\frac{\gamma}{2}[\kappa(\vartheta_{2m},s)+\kappa(F_2r,s)]$$
$$\frac{2-\gamma}{2}\kappa(s,\ F_2r)\leq b_m+a_m+\frac{\gamma}{2}b_{2m}$$

i.e.
$$\kappa(s,\ F_2r)\leq\frac{2}{2-\gamma}[b_m+a_m+\frac{\gamma}{2}b_m]$$
$$=\frac{2}{2-\gamma}a_m+\frac{2+\gamma}{2-\gamma}b_m.$$

(iii) If $H_{\varsigma_{2m},r}(F_1,F_2,F_3,F_4)=\frac{1}{2}[\kappa(\vartheta_{2m},s)+\kappa(\vartheta_{2m+1},s)]$. Then we have

$$\kappa(s,\ F_2r)\leq b_m+a_m+\frac{\gamma}{2}[\kappa(\vartheta_{2m},s)+\kappa(\vartheta_{2m+1},s)]$$
$$\leq b_m+a_m+\frac{\gamma}{2}[b_{2m}+b_{2m+1}]$$
$$\leq a_m+b_m(1+\gamma).$$

(iv) If $H_{\varsigma_{2m},r}(F_1,F_2,F_3,F_4)=\frac{1}{2}[\kappa(F_2r,s)+\kappa(\vartheta_{2m+1},s)]$. Then we have

$$\kappa(s,\ F_2r)\leq b_m+a_m+\frac{\gamma}{2}[\kappa(F_2r,s)+\kappa(\vartheta_{2m+1},s)]$$
$$\frac{2-\gamma}{2}\kappa(s,\ F_2r)\leq b_m+a_m+\frac{\gamma}{2}b_{2m+1}$$

i.e.
$$\kappa(s,\ F_2r)\leq\frac{2}{2-\gamma}[a_m+\frac{\gamma+2}{2}b_m]$$

$$= \frac{2}{2-\gamma} a_m + \frac{2+\gamma}{2-\gamma} b_m.$$

Implies $\kappa(s,\ F_2 r) = 0$, i.e. $F_2 r = s$. Thus $F_2 r = F_4 r = s$. Consequently $\{F_1, F_3\}$ and $\{F_2, F_4\}$ have a common point of coincidence from W. Since the pairs $\{F_1, F_3\}$ and $\{F_2, F_4\}$ are WC, it follows that

$$F_1 s = F_1 F_3 g = F_3 F_1 g = F_3 s = m_1 \text{ (say)}, \qquad (38)$$

and also

$$F_2 s = F_2 F_4 r = F_4 F_2 r = F_4 s = m_2 \text{ (say)}. \qquad (39)$$

Now

$$\kappa(m_1, m_2) = \kappa(F_1 s, F_2 s) \preceq \gamma H_{s,s}(F_1, F_2, F_3, F_4),$$

where

$$H_{s,s}(F_1, F_2, F_3, F_4) \in \left\{\frac{1}{2}[\kappa(F_3 s, F_4 s) + \kappa(F_1 s, F_3 s)], \frac{1}{2}[\kappa(F_3 s, F_4 s) + \kappa(F_2 s, F_4 s)], \frac{1}{2}[\kappa(F_3 s, F_4 s) + \kappa(F_1 s, F_4 s)], \frac{1}{2}[\kappa(F_2 s, F_4 s), \kappa(F_1 s, F_4 s)]\right\}$$

$$= \left\{\frac{1}{2}[\kappa(F_3 s, F_4 s) + \kappa(m_1, m_2)], \frac{1}{2}[\kappa(F_3 s, F_4 s)\ \kappa(m_1, m_2)], \frac{1}{2}[\kappa(F_3 s, F_4 s) + \kappa(F_1 s, F_4 s)], \frac{1}{2}[\kappa(m_1, m_2), \kappa(F_1 s, F_4 s)]\right\} \text{ (using (38) and (39))}$$

$$= \{\frac{1}{2}\ \kappa(F_1 s, F_2 s),\ \ \kappa(F_1 s, F_2 s)\}.$$

If $H_{s,s}(F_1, F_2, F_3, F_4)$ is equal to $\frac{1}{2}\ \kappa(F_1 s, F_2 s)$ or $\kappa(F_1 s, F_2 s)$, in both the cases we conclude that $\kappa(m_1, m_2) = 0$ i.e. $m_1 = m_2$. Thus $F_1 s = F_2 s = F_3 s = F_4 s$. Next, we shall show that $F_2 s = s$. We have

$$\kappa(s, F_2 s) = \kappa(F_1 g, F_2 s) \preceq \gamma H_{g,s}(F_1, F_2, F_3, F_4),$$

where

$$H_{g,s}(F_1, F_2, F_3, F_4) \in \left\{\frac{1}{2}[\kappa(F_3 g, F_4 s) + \kappa(F_1 g, F_3 g)], \frac{1}{2}[\kappa(F_3 g, F_4 s) + \kappa(F_2 s, F_4 s)], \frac{1}{2}[\kappa(F_3 g, F_4 s) + \kappa(F_1 g, F_4 s)], \frac{1}{2}[\kappa(F_2 s, F_4 s), \kappa(F_1 g, F_4 s)]\right\}$$

$$= \{\frac{1}{2}\kappa(s, F_2 s),\ \ \kappa(s, F_2 s)\}.$$

If $H_{g,s}(F_1, F_2, F_3, F_4)$ is equal to $\frac{1}{2}\kappa(s, F_2 s)$ or $\kappa(s, F_2 s)$, Then clearly we have $\kappa(s,\ F_2 s) = 0$. Hence $s = F_2 s = F_4 s$ i.e. $F_1 s = F_2 s = F_3 s = F_4 s = s$. Thus s is a c.f.p. of F_1, F_2, F_3 & F_4. Next we shall show the uniqueness of s. Assume some other c.f.p s^* of F_1, F_2, F_3 & F_4. So we write $F_1 s^* = F_2 s^* = F_3 s^* = F_4 s^* = s^*$. Then

$$\kappa(s, s^*) = \kappa(F_1 s, F_2 s^*) \preceq \gamma H_{s,s^*}(F_1, F_2, F_3, F_4).$$

where

$$H_{s,s^*}(F_1, F_2, F_3, F_4) \in \left\{\frac{1}{2}[\kappa(F_3 g, F_4 s^*) + \kappa(F_1 g, F_3 g)], \frac{1}{2}[\kappa(F_3 g, F_4 s^*) + \kappa(F_2 s^*, F_4 s^*)], \frac{1}{2}[\kappa(F_3 g, F_4 s^*) + \kappa(F_1 g, F_4 s^*)], \frac{1}{2}[\kappa(F_2 s^*, F_4 s^*), \kappa(F_1 g, F_4 s^*)]\right\}$$

$$= \left\{0, \frac{1}{2}\kappa(s, s^*), \kappa(s, s^*)\right\}.$$

If $H_{s,s^*}(F_1, F_2, F_3, F_4)$ is equal to $\frac{1}{2}\kappa(s, s^*)$ or $\kappa(s, s^*)$, Then clearly we have $\kappa(s, s^*) = 0$ i.e. $s = s^*$. Hence s is a c.f.p of F_1, F_2, F_3 & F_4 which is unique. The proofs for the cases in which $F_1(W), F_2(W)$ and $F_4(W)$ is complete are similar.

Corollary 2.5. Let (W, κ, Q) be a vector valued r.m.s with Q-Archimedean. Suppose that F_1, F_2 and F_3 are self-mappings on W, with $F_1(W) \cup F_2(W) \subseteq F_3(W)$. Let the following condition hold:

$$\kappa(F_1 \varsigma, F_2 \vartheta) \preceq\ \gamma H_{\varsigma,\vartheta}(F_1, F_2, F_3)$$
$$\kappa(F_3 \varsigma, F_2 \vartheta) \preceq\ \gamma H_{\varsigma,\vartheta}(F_1, F_2, F_3)$$

for all $\varsigma, \vartheta \in W$ and $\gamma \in [0, 1)$, where

$$H_{\varsigma,\vartheta}(F_1, F_2, F_3) \in \left\{\frac{1}{2}[\kappa(F_3 \varsigma, F_3 \vartheta) + \kappa(F_1 \varsigma, F_3 \varsigma)], \frac{1}{2}[\kappa(F_3 \varsigma, F_3 \vartheta) + \kappa(F_2 \vartheta, F_3 \vartheta)], \frac{1}{2}[\kappa(F_3 \varsigma, F_3 \vartheta) + \kappa(F_1 \varsigma, F_3 \vartheta)], \frac{1}{2}[\kappa(F_2 \vartheta, F_3 \vartheta), \kappa(F_1 \varsigma, F_3 \vartheta)]\right\}.$$

Let the pairs $\{F_1, F_3\}$ and $\{F_2, F_3\}$ are given to be WC and if one of $F_1(W), F_2(W)$ or $F_3(W)$ is Q-complete subspace of W then F_1, F_2 and F_3 have c.f.p which is unique.
Proof. Result follows by taking mapping $F_3 = F_4$ in Theorem 2.4.

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