Harmonic Mean Cordial Labeling of One Chord $C_n \vee G$

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Abstract
All the graphs considered in this article are simple and undirected. Let $G = (V(G),E(G))$ be a simple undirected graph. A function $f : V(G) \to \{1,2\}$ is called Harmonic Mean Cordial if the induced function $f^* : E(G) \to \{1,2\}$ defined by $f^*(uv) = \left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$ for any $i, j \in \{1,2\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and number of edges with label x respectively and $\lfloor x \rfloor$ denotes the greatest integer less than or equals to x. A graph G is called a harmonic mean cordial graph if it admits harmonic mean cordial labeling. In this article, we have discussed the harmonic mean cordial labeling of One Chord $C_n \vee G$.

Keywords: Harmonic Mean Cordial Labeling, Complete graph, Cycle, One Chord Cycle, Join of two graphs.
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1. INTRODUCTION

We begin with simple, finite, connected and undirected graph $G = (V(G),E(G))$. For terminology and notation not defined here we follow Balakrishnan and Ranganathan [1].

In [2] J. Gowri and J. Jayapriya defined Harmonic Mean Cordial labeling of graph G. Let $G = (V(G), E(G))$ be a simple undirected Graph. A function $f : V(G) \to \{1,2\}$ is called Harmonic Mean Cordial if the induced function $f^* : E(G) \to \{1,2\}$ defined by $f^*(uv) = \left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$ for any $i, j \in \{1,2\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and number of edges with label x respectively and $\lfloor x \rfloor$ denotes the greatest integer less than or equals to x. A graph G is called a harmonic mean cordial graph if it admits harmonic mean cordial labeling. For the sake of convenience of the reader we use 'HMC' for harmonic mean cordial labeling and '$C_{(1,n-1)}$' for One Chord Cycle Graph. It is useful to recall some useful definitions of graph theory to make this article self-contained. Motivated by the interesting results proved in [3, 4, 5] and on Root Cube Mean Cordial Labeling in [6], we have discussed HMC labeling of Harmonic Mean Cordial labeling of One Chord $C_n \vee G$.

Definition 1 [7] A Chord of a cycle C_n is an edge not in C_n whose endpoints lie in C_n.
Definition 2 [1] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then union of G_1 and G_2 is denoted by $G_1 \cup G_2$ is the graphs whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$. When G_1 and G_2 are vertex disjoint $G_1 \cup G_2$ is called sum of G_1 and G_2 and it is denoted by $G_1 + G_2$.
Definition 3 [1] Let G_1 and G_2 be two vertex disjoint graphs. Then the join $G_1 \vee G_2$ of G_1 and G_2 is the super graph of G_1+G_2 in which each vertex of G_1 is also adjacent to every vertex of G_2.

In Theorem 2.1, we have proved that the complete graph $K_n \vee C_{(1,m-1)}$ is not HMC for any $n,m \ge 2$ and $n,m \in \mathbb{N}$.
In Theorem 2.2, we have proved that $C_{(1,m-1)} \vee C_{(1,n-1)}$ is not HMC for any $n,m \ge 2$ and $n,m \in \mathbb{N}$.

2. MAIN RESULTS

Proposition 2.1.
$K_n \vee C_{(1,n-1)}$ is not HMC for $n \ge 2$.

Proof:

Suppose that $K_n \vee C_{(1,n-1)}$ is HMC. Note that, $|V(K_n \vee C_{(1,n-1)})| = 2n$ and $|E(K_n \vee C_{(1,n-1)})| = n\frac{(n-1)}{2} + n + n^2 + 1$. Since, $|V(K_n \vee C_{(1,n-1)})| = 2n$ and we have assume that $K_n \vee C_{(1,n-1)}$ is HMC. We have $v_f(1) = v_f(2) = n$.
Case 1: All the vertices of label 1 and label 2 are in sequence in $C_{(1,n-1)}$
Suppose that we have r number of vertices with label 1 in K_n. So, we have $(n - r)$ vertices of of label 1 in $C_{(1,n-1)}$. Hence, we have $(n - r)$ vertices of label 2 in K_n and r vertices of label 2 in $C_{(1,n-1)}$. Note that, $e_f(1) = (n-r)r + r\frac{(r-1)}{2} + (n-r)^2 + (n-r+1) + nr + 1$ and $e_f(2) = \frac{(n-r)(n-r-1)}{2} + r(n-r) + (r-1)$.

Now, $e_f(1) - e_f(2) = \frac{n^2}{2} + r^2 + \frac{3n}{2} - 3r + 3$. If $r \geq 3$ then as $n \geq 4$, we have $e_f(1) - e_f(2) > 1$.

If $r = 1$ and $r = 2$ then $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{3n}{2} + 1 > 1$. So, $e_f(1) - e_f(2) > 1$.

Case 2: Some of the vertices of label 2 are not in sequence in $C_{(1, n-1)}$

Suppose that we have r number of vertices with label 1 in K_n. So, we have $(n - r)$ vertices of label 1 in $C_{(1,n-1)}$. Hence, we have $(n - r)$ vertices of label 2 in K_n and r vertices of label 2 in $C_{(1,n-1)}$. Suppose that there exist j number of vertices with label 2 are not in sequence in $C_{(1,n-1)}$. Then, we have $e_f(1) = \frac{r(r-1)}{2} + r(n-r) + rn + (n-r)^2 + (n - r + j + 1)$ and $e_f(2) = (r - j - 1) + \frac{(n-r-1)(n-r)}{2} + r(n-r)$ Now, $e_f(2)$ in case 2 ≤ $e_f(2)$ in case 1 and $e_f(1)$ in case 2 ≥ $e_f(1)$ in case 1. So, $e_f(1) - e_f(2)$ in this case ≥ $e_f(1) - e_f(2)$ in case 1. Now, we have already proved in case 1 that $e_f(1) - e_f(2) > 1$.

Case 3: We have *n* number of vertices with label 1 in K_n and *n* number of vertices with label 2 in $C_{(1, n-1)}$

Then, we have $e_f(1) = \frac{n(n-1)}{2} + n^2$ and $e_f(2) = n+1$. Then, $e_f(1) - e_f(2) = \frac{n(n-1)}{2} + n^2 - n - 1 = \frac{3n^2}{2} - \frac{3n}{2} - 1 > 1$ as $n^2 > n$.

Case 4: We have *n* number of vertices with label 2 in K_n and *n* number of vertices with label 1 in $C_{(1, n-1)}$

Then we have, $e_f(1) = n^2+n+1$ and $e_f(2) = \frac{n(n-1)}{2}$. Then, $e_f(1) - e_f(2) = n^2 + n + 1 - \frac{n(n-1)}{2} = \frac{n^2}{2} + \frac{3n}{2} + 1 > 1$. Hence, $K_n \vee C_{(1,n-1)}$ is not HMC.

Proposition 2.2.

$K_n \vee C_{(1,m-1)}$ is not HMC, where $m + n$ is even and $m,n \geq 2$.

Proof:

Note that, $|V(K_n \vee C_{(1,m-1)})| = m + n$. Suppose that $K_n \vee C_{(1,m-1)}$ is Harmonic mean cordial.

Then we have, $|v_f(1)| = \frac{m+n}{2} = |v_f(2)|$.

Case 1: All the vertices with label 1 and label 2 are in sequence in $C_{(1,m-1)}$

Suppose that we have r number of vertices with label 1 in K_n. So, we have $(\frac{m+n}{2} - r)$ vertices with label 1 in $C_{(1,m-1)}$. Hence, we have $(n-r)$ vertices with label 2 in K_n and $m - (\frac{m+n}{2} - r) = (\frac{m-n}{2} + r)$ vertices with label 2 in $C_{(1,m-1)}$. Then we have, $e_f(1) = \frac{r(r-1)}{2} + rm + (n-r)(\frac{m+n}{2} - r) + (\frac{m+n}{2} - r + 1) + r(n-r)$ and $e_f(2) = \frac{(n-r)(n-r-1)}{2} + (n-r)(\frac{m-n}{2} + r) + (\frac{m-n}{2} + r - 1) + 1$.

Then, $e_f(1) - e_f(2) = mr + \frac{n^2}{2} - nr + \frac{3n}{2} + r^2 - 3r + 1 = (r-n)^2(\frac{1}{2}) + \frac{r^2}{2} + \frac{3n}{2} + 2 + r(m-3) - 1$.

Now, $n > r$. So, $e_f(1) - e_f(2) > 1$.

Case 2: Some of the vertices with label 2 are not in sequence in $C_{(1,m-1)}$

Suppose that we have r number of vertices with label 1 in K_n. So, we have $(\frac{m+n}{2} - r)$ vertices with label 1 in $C_{(1,m-1)}$. Hence, we have $(n - r)$ vertices with label 2 in K_n and $m - (\frac{m+n}{2} - r) = (\frac{m-n}{2} + r)$ vertices with label 2 in $C_{(1,m-1)}$. Suppose that there exist j number of vertices from $(\frac{m-n}{2} + r)$ with label 2 are not in sequence in $C_{(1,m-1)}$. Then we have, $e_f(1) = \frac{r(r-1)}{2} + r(n-r) + rm + (n-r)(\frac{m+n}{2} - r) + (\frac{m+n}{2} - r + j + 1)$ and $e_f(2) = \frac{(n-r)(n-r-1)}{2} + (n-r)(\frac{m-n}{2} + r) + (\frac{m-n}{2} + r - j)$. Now, $e_f(2)$ in case 2 ≤ $e_f(2)$ in case 1 and $e_f(1)$ in case 2 ≥ $e_f(1)$ in case 1. So, $e_f(1)-e_f(2)$ in this case ≥ $e_f(1)-e_f(2)$ in case 1. Now, we have already proved in case 1 that $e_f(1) - e_f(2) > 1$.

Case 3: $m < n$

Subcase 3.1: All the vertices in $C_{(1,m-1)}$ are with label 1

Suppose that we have r number of vertices with label 1 in K_n. So, we have $(n - r)$ vertices with label 2 in K_n. Then we have, $e_f(1) = \frac{r(r-1)}{2} + mn + m + r(n-r) + 1$ and $e_f(2) = \frac{(n-r)(n-r-1)}{2}$.

Then, $e_f(1) - e_f(2) = mn + m + 2nr + \frac{n}{2} - r^2 - r - \frac{n^2}{2}$. We know that, $r = \frac{m+n}{2}$.

Then, $e_f(1) - e_f(2) = \frac{3mn}{2} + \frac{m}{2} + (\frac{n^2}{4} - \frac{m^2}{4}) + 1$. We know that $n > m$. So, $e_f(1) - e_f(2) > 1$.

Subcase 3.2: All the vertices in $C_{(1,m-1)}$ are with label 2

Suppose that we have r number of vertices with label 1 in K_n. So, we have $(n - r)$ vertices with label 2 in K_n. Then we have, $e_f(1) = \frac{r(r-1)}{2} + rm + r(n-r)$ and $e_f(2) = \frac{(n-r)(n-r-1)}{2} + m(n - r) + m + 1$.

Then, $e_f(1)-e_f(2)=m(n-r)+mr-m-\frac{n^2}{2}+2nr+\frac{n}{2}-r^2-r-1$. We know that, $r=\frac{m+n}{2}$. Then, $e_f(1)-e_f(2)=\frac{mn}{2}+\frac{3m^2}{4}+\frac{n^2}{4}-\frac{3m}{2}-1$ as $m\geq 2$. So, $e_f(1) - e_f(2) > 1$.

Case 4: *m > n*

Subcase 4.1: All the vertices in K_n are with label 1

Suppose that we have r number of vertices with label 1 in $C_{(1,m-1)}$. So, we have $(m - r)$ vertices with label 2 in $C_{(1,m-1)}$.

Subsubcase 4.1.1: All the vertices with label 2 are in sequence in $C_{(1,m-1)}$

Then we have, $e_f(1)=\frac{n(n-1)}{2}+(r+1)+nm$ and $e_f(2) = m - r$. Then, $e_f(1) - e_f(2) = \frac{n(n-1)}{2}+(r+1)+nm-m+r$. We know that, $nm > m$. So, $e_f(1) - e_f(2) > 1$.

Subsubcase 4.1.2: Some of the vertices with label 2 are not in sequence in $C_{(1,m-1)}$

Suppose that we have j number of vertices with label 2 are not in sequence in $C_{(1,m-1)}$. Suppose that j number of vertices are not j sequence. Then we have, $e_f(1)=\frac{n(n-1)}{2}+nm+r+j$ and $e_f(2) = m - r - j + 1$. Now, $e_f(2)$ in subsubcase 4.1.2 ≤ $e_f(2)$ in subsubcase 4.1.1 and $e_f(1)$ in subsubcase 4.1.2 ≥ $e_f(1)$ in subsubcase 4.1.1. So, $e_f(1) - e_f(2)$ in this case ≥ $e_f(1) - e_f(2)$ in subsubcase 4.1.1. Now, we have already proved in subsubcase 4.1.1 that $e_f(1) - e_f(2) > 1$.

Subcase 4.2: All the vertices in K_n are with label 2

Suppose that we have r number of vertices with label 1 in $C_{(1,m-1)}$. So, we have $(m - r)$ vertices with label 2 in $C_{(1,m-1)}$.

Subsubcase 4.2.1: All the vertices with label 2 are in sequence in $C_{(1,m-1)}$

Then we have, $e_f(1) = (r + 1) + nr + 1$ and $e_f(2)=\frac{n(n-1)}{2}+(m-r-1)+n(m-r)$. Then, $e_f(1)-e_f(2)=(r+1)+rn+1-\frac{n(n-1)}{2}-m+r+1-mn+rn$. We know that, $r=\frac{m+n}{2}$. Then, $e_f(1)-e_f(2)=\frac{n^2}{2}+\frac{3n}{2}+3>1$.

Subsubcase 4.2.2: Suppose that some of the vertices with label 2 are not in sequence in $C_{(1,m-1)}$

Then we have, $e_f(1) = m + rn$ and $e_f(2)=\frac{n(n-1)}{2}+(m-r)n+1$. Now, $e_f(2)$ in subsubcase 4.2.2 ≤ $e_f(2)$ in subsubcase 4.2.1 and $e_f(1)$ in subsubcase 4.2.2 ≥ $e_f(1)$ in subsubcase 4.2.1. So, $e_f(1) - e_f(2)$ in this case is ≥ $e_f(1) - e_f(2)$ in subsubcase 4.2.1. Now, we have already proved in subsubcase 4.2.1 that $e_f(2) - e_f(2) > 1$. Hence, $e_f(2) - e_f(2) > 1$ in this case.

Hence, $K_n \vee C_{(1,m-1)}$ is not HMC, where $m + n$ is even and $m,n \geq 2$.

Proposition 2.3.

$K_n \vee C_{(1,m-1)}$ is not HMC, where $m + n$ is odd and $m,n \geq 2$.

Proof:

Note that, $|V(K_n \vee C_{(1,m-1)})| = m + n$. Suppose that $K_n \vee C_{(1,m-1)}$ is HMC.

Case 1: All the vertices with label 1 and label 2 in C_m are in sequence in $C_{(1,m-1)}$

In this case we have two possibilities (i) $v_f(1)=\frac{m+n+1}{2}$ and $v_f(2)=\frac{m+n-1}{2}$ (ii) $v_f(1)=\frac{m+n-1}{2}$ and $v_f(2)=\frac{m+n+1}{2}$. So, we consider the following cases.

Subcase 1.1: $v_f(1)=\frac{m+n+1}{2}$ and $v_f(2)=\frac{m+n-1}{2}$

Suppose that we have r number of vertices with label 1 in K_n. So, we have $\left(\frac{m+n+1}{2}-r\right)$ vertices of label 1 in $C_{(1,m-1)}$. Hence, we have $(n-r)$ vertices with label 2 in K_n and $m-\left(\frac{m+n+1}{2}-r\right)=\left(\frac{m-n-1}{2}+r\right)$ vertices with label 2 in $C_{(1,m-1)}$. Then we have, $e_f(1)=\frac{r(r-1)}{2}+rn+r(n-r)+\frac{m+n+1}{2}-r+1+(n-r(\frac{m+n+1}{2}-r)+1$ and $e_f(2)=\frac{(n-r)(n-r-1)}{2}+(n-r)(\frac{m-n-1}{2}+r)+\frac{m-n-1}{2}+r-1$.

Then, $e_f(1)-e_f(2)=\frac{n^2}{2}+\frac{5n}{2}+r^2-4r+4>1$ as $n>r$.

Subcase 1.2: $v_f(1)=\frac{m+n-1}{2}$ and $v_f(2)=\frac{m+n+1}{2}$

Suppose that we have r number of vertices with label 1 in K_n. So, we have $\left(\frac{m+n-1}{2}-r\right)$ vertices of label 1 in $C_{(1,m-1)}$. Hence, we have $(n-r)$ vertices with label 2 in K_n and $m-\left(\frac{m+n-1}{2}-r\right)=\left(\frac{m-n+1}{2}+r\right)$ vertices with label 2 in $C_{(1,m-1)}$. Then we have, $e_f(1)=\frac{r(r-1)}{2}+rm+r(n-r)+\frac{m+n-1}{2}-r+1+(n-r)(\frac{m+n-1}{2}-r)$ and $e_f(2)=\frac{(n-r)(n-r-1)}{2}+(n-r)(\frac{m-n+1}{2}+r)+\frac{m-n+1}{2}+r$.

Then, $e_f(1)-e_f(2)=r^2+\frac{n^2}{2}+\frac{n}{2}+mr-2r-nr>1$ as $n\geq 2$.

Case 2: Some of the vertices with label 2 are not in sequence in $C_{(1,m-1)}$

Subcase 2.1: Suppose that $v_f(1) = \frac{m+n+1}{2}$ **and** $v_f(2) = \frac{m+n-1}{2}$

Suppose that we have *r* number of vertices with label 1 in K_n. So, we have $(\frac{m+n+1}{2} - r)$ vertices of label 1 in $C_{(1,m-1)}$. Hence, we have $(n-r)$ vertices with label 2 in K_n and $m - (\frac{m+n+1}{2} - r) = (\frac{m-n-1}{2} + r)$ vertices with label 2 in $C_{(1,m-1)}$. Suppose that there exist *j* number of vertices from $(\frac{m-n-1}{2} + r)$ with label 2 are not in sequence in $C_{(1,m-1)}$. Then we have, $e_f(1) = \frac{r(r-1)}{2} + rm + r(n-r) + (\frac{m+n+1}{2} - r + j + 1) + (n-r)(\frac{m+n+1}{2} - r)$ and $e_f(2) = (\frac{m-n-r}{2} - r - j - 1) + \frac{(n-r)(n-r-1)}{2} + (n-r)(\frac{m-n-1}{2} + r)$. Now, $e_f(2)$ in subcase 2.1 ≤ $e_f(2)$ in subcase 1.1 and $e_f(1)$ subcase 2.1 ≥ $e_f(1)$ in subcase 1.1. So, $e_f(1) - e_f(2)$ in this case ≥ $e_f(1) - e_f(2)$ in subcase 1.1. Now, we have already proved in subcase 1.1 that $e_f(1) - e_f(2) > 1$.

Subcase 2.2: $v_f(1) = \frac{m+n-1}{2}$ **and** $v_f(2) = \frac{m+n+1}{2}$

Suppose that we have *r* number of vertices with label 1 in K_n. So, we have $(\frac{m+n-1}{2} - r)$ vertices of label 1 in $C_{(1,m-1)}$. Hence, we have $(n-r)$ vertices with label 2 in K_n and $m - (\frac{m+n-1}{2} - r) = (\frac{m-n+1}{2} + r)$ vertices with label 2 in $C_{(1,m-1)}$. Suppose that there exist *j* number of vertices from $(\frac{m-n+1}{2} + r)$ with label 2 are not in sequence in $C_{(1,m-1)}$. Then we have, $e_f(1) = \frac{r(r-1)}{2} + rm + r(n-r) + (\frac{m+n-1}{2} - r + j + 1) + (n-r)(\frac{m+n-1}{2} - r)$ and $e_f(2) = (\frac{m-n+1}{2} + r - j - 1) + \frac{(n-r)(n-r-1)}{2} + (n-r)(\frac{m-n+1}{2} + r)$. Now, $e_f(2)$ in subcase 2.2 ≤ $e_f(2)$ in subcase 1.2. and $e_f(1)$ subcase 2.2 ≥ $e_f(1)$ in subcase 1.2. So, $e_f(1) - e_f(2)$ in this case ≥ $e_f(1) - e_f(2)$ in subcase 2.1. Now, we have already proved in subcase 2.1 that $e_f(1) - e_f(2) > 1$.

Case 3: $m < n$

Subcase 3.1: All the vertices in $C_{(1,m-1)}$ are with label 1 and some vertices with label 1 are in K_n

Suppose that there exist *r* number of vertices with label 1 in K_n. So, there exists $(n - r)$ vertices with label 2 in K_n. Suppose that we have *m* number of vertices with label 1 in $C_{(1,m-1)}$. Then we have, $e_f(1) = \frac{r(r-1)}{2} + r(n-r) + mn + m + 1$ and $e_f(2) = \frac{(n-r)(n-r-1)}{2}$. Then, $e_f(1) - e_f(2) = mn + m + 2nr + 1 + \frac{n}{2} - r - r^2 - \frac{n^2}{2}$.

In this case we have two possibilities

(i) $m + r = \frac{m+n+1}{2}$

(ii) $m + r = \frac{m+n-1}{2}$

So, we consider the following cases.

Subsubcase 3.1.1: $m + r = \frac{m+n+1}{2}$

Therefore, $r = \frac{n-m+1}{2}$. Then, $e_f(1) - e_f(2) = \frac{mn}{2} + (2m - \frac{3}{4}) + (\frac{n^2}{4} - \frac{m^2}{4}) + \frac{n}{2} > 1$ as $m < n$ and $2m > \frac{3}{4}$ as $m \geq 2$.

Subsubcase 3.1.2: $m + r = \frac{m+n-1}{2}$

Thenrefore, $r = \frac{n-m-1}{2}$. Then, $e_f(1) - e_f(2) = (\frac{mn}{2} - \frac{n}{2}) + m + (\frac{n^2}{4} - \frac{m^2}{4}) + \frac{1}{4} > 1$ as $n > m$.

Subcase 3.2: All the vertices in $C_{(1,m-1)}$ are with label 2 and some vertices with label 2 are in K_n

Suppose that there exist *r* numbers of vertices with label 1 in K_n. So, there exists $(n - r)$ vertices with label 2 in K_n. Suppose that we have *m* number of vertices with label 2 in $C_{(1,m-1)}$. Then we have, $e_f(1) = \frac{r(r-1)}{2} + r(n-r) + rm$ and $e_f(2) = \frac{(n-r)(n-r-1)}{2} + m(n-r) + m$. Then, $e_f(1) - e_f(2) = 2mr - mn - \frac{n^2}{2} + 2nr + \frac{n}{2} - r^2 - r - m$.

Subsubcase 3.2.1: $r = \frac{m+n+1}{2}$

Then, $e_f(1) - e_f(2) = \frac{3m^2}{4} + (\frac{mn}{2} - m) + (\frac{n^2}{4} - \frac{3}{4}) + \frac{n}{2} > 1$ as $m, n \geq 2$.

Subsubcase 3.2.2: $r = \frac{m+n-1}{2}$

Then, $e_f(1) - e_f(2) = \frac{3m^2}{4} + \frac{mn}{2} - 2m + \frac{n^2}{4} + \frac{1}{4} - \frac{n}{2} = (\frac{n^2}{4} - \frac{n}{2}) + m(\frac{3m}{4} + \frac{n}{2} - 2) + \frac{1}{4} > 1$ as *m,n* ≥ 2.

Case 4: $m > n$ and all the vertices with label 2 are in sequence in $C_{(1,m-1)}$

Subcase 4.1: All the vertices in K_n are with label 1 and some vertices with label 1 are in $C_{(1,m-1)}$

Suppose that there exist r number of vertices with label 1 in $C_{(1,m-1)}$. So, there exists $(m - r)$ vertices with label 2 in $C_{(1,m-1)}$. Suppose that we have n number of vertices with label 1 in K_n.

Then we have, $e_f(1) = mn + (r+1) + n\frac{(n-1)}{2}$ and $e_f(2) = m - r$. Then, $e_f(1) - e_f(2) = (mn - m) + 2r + (\frac{n^2}{2} - \frac{n}{2}) + 1 > 1$ as $mn > m$ and $\frac{n^2}{2} > \frac{n}{2}$, where, $m,n \geq 2$.

Subcase 4.2: All the vertices in K_n are with label 2 and some vertices with label 2 are in $C_{(1,m-1)}$

Suppose that there exist r number of vertices with label 1 in $C_{(1,m-1)}$. So, there exists $(m - r)$ vertices with label 2 in $C_{(1,m-1)}$. Suppose that we have n number of vertices with label 2 in K_n.

Then we have, $e_f(1) = rn + (r + 1) + 1$ and $e_f(2) = \frac{n(n-1)}{2} + n(m-r) + (m-r-1)$. Then, $e_f(1) - e_f(2) = 2r + 2nr - \frac{n^2}{2} + \frac{n}{2} - mn - m + 3$.

Subsubcase 4.2.1: $r = \frac{m+n+1}{2}$

Then, $e_f(1) - e_f(2) = \frac{5n}{2} + \frac{n^2}{2} + 4 > 1$.

Subsubcase 4.2.2: $r = \frac{m+n-1}{2}$

Then, $e_f(1) - e_f(2) = \frac{n^2}{2} + \frac{n}{2} + 2 > 1$.

Case 5: $m > n$ and some of the vertices with label 2 are not in sequence in $C_{(1,m-1)}$

Subcase 5.1:All the vertices in K_n are with label 1 and some vertices with label 1 are in $C_{(1,m-1)}$

Suppose that there exist r number of vertices with label 1 in $C_{(1,m-1)}$. So, there exists $(m - r)$ vertices with label 2 in $C_{(1,m-1)}$. Suppose that we have n number of vertices with label 1 in K_n. Suppose that we have j number of vertices with label 2 are not in sequence in $C_{(1,m-1)}$. Then, $e_f(1) = \frac{n(n-1)}{2} + mn + (r+j+1)$ and $e_f(2) = m - r - j$. Now, $e_f(2)$ in subcase 5.1 ≤ $e_f(2)$ in subcase 4.1 and $e_f(1)$ in subsubcase 5.1 ≥ $e_f(1)$ in subsubcase 4.1. So, $e_f(1)-e_f(2)$ in this case is ≥ $e_f(1)-e_f(2)$ in subcase 4.1. Now, we have already proved in subcase 4.1 that $e_f(1)-e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this case.

Subcase 5.2: All the vertices in K_n are with label 2 and some vertices with label 2 are in $C_{(1,m-1)}$

Suppose that there exist r number of vertices with label 1 in $C_{(1,m-1)}$. So, there exists $(m - r)$ vertices with label 2 in $C_{(1,m-1)}$. Suppose that we have n number of vertices with label 2 in K_n. Suppose that we have j number of vertices with label 2 are not in sequence in $C_{(1,m-1)}$. Then $e_f(1) = nr + (r + j + 1)$ and $e_f(2) = \frac{n(n-1)}{2} + mn + m - nr - r - j$. Now, $e_f(2)$ in subcase 5.2 ≤ $e_f(2)$ in subcase 4.2 and $e_f(1)$ in subsubcase 5.2 ≥ $e_f(1)$ in subsubcase 4.2. So, $e_f(1) - e_f(2)$ in this case is ≥ $e_f(1) - e_f(2)$ in subcase 4.2. Now, we have alredy proved in subcase 4.2 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this case.

Hence, $K_n \vee C_{(1,m-1)}$ is not HMC, where $m + n$ is odd and $m,n \geq 2$.

Theorem 2.1. $K_n \vee C_{(1,m-1)}$ is not HMC, where $m,n \geq 2$, $m,n \in \mathbb{N}$.

Proof:

Proof follows from Propositions 2.1, 2.2 and 2.3.

Proposition 2.4.

$\boldsymbol{C_{(1,m-1)}} \vee \boldsymbol{C_{(1,n-1)}}$ is not HMC, where $m = n$ and $m \geq 2$.

Proof:

Suppose that $C_{(1,m-1)} \vee C_{(1,n-1)}$ is HMC for $m = n$. Note that, $|V(C_{(1,m-1)} \vee C_{(1,n-1)})| = 2n$ and $|E(C_{(1,m-1)} \vee C_{(1,n-1)})| = n + m + nm + 2 = 2 + 2n + n^2$ as $n = m$. Since, $|V(C_{(1,m-1)} \vee C_{(1,m-1)})| = m+n = 2n$ as $n = m$. We have assume that $C_{(1,m-1)} \vee C_{(1,n-1)}$ is HMC for $n = m$. We have $v_f(1) = v_f(2) = n$.

Case 1: All the vertices of label 1 are in sequence in $C_{(1,m-1)}$ and $C_{(1,n-1)}$

Then, it is clear that all the vertices of label 2 are in sequence in $C_{(1,m-1)}$ and $C_{(1,n-1)}$. Suppose that we have r number of vertices with label 1 in $C_{(1,m-1)}$. So, we have $(n - r)$ vertices of of label 1 in $C_{(1,n-1)}$. Hence, we have $(m - r)$ vertices of label 2 in $C_{(1,m-1)}$ and r vertices of label 2 in $C_{(1,n-1)}$. Note that, $e_f(1) = (r + 1) + (n - r + 1) + rn + (n - r)n + 1$ and $e_f(2) = (n - r - 1) + (r - 1) + r(n - r) + 1$. Then, $e_f(1) - e_f(2) = 2n + 2 + rn + n^2 - r^2$. We know that, $n > r$. So, $e_f(1) - e_f(2) > 1$.

Case 2: Some of the vertices of label 2 are not in sequence in $C_{(1,m-1)}$ and $C_{(1,n-1)}$

Suppose that we have r number of vertices with label 1 in $C_{(1,m-1)}$. So, we have $(n - r)$ vertices of label 1 in $C_{(1,n-1)}$. Hence, we have $(m - r)$ vertices of label 2 in $C_{(1,m-1)}$ and r vertices of label 2 in $C_{(1,n-1)}$. Suppose that there exist i number of vertices with label 2 are not in sequence in $C_{(1,m-1)}$ and j number of vertices with label 2 are not in sequence in $C_{(1,n-1)}$. Note that, $e_f(1) = (r+i+1)+(n-r+j+1)+rn+(n-r)m+1$ and $e_f(2) = (n-r-i-1)+(r-j-1)+r(n-r)+1$. Now, $e_f(2)$ in case 2 ≤ $e_f(2)$ in case 1 and $e_f(1)$ in case 2 ≥ $e_f(1)$ in case 1. So, $e_f(1)$

− $e_f(2)$ in this case is ≥ $e_f(1) - e_f(2)$ We have already proved in case 1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this case.

Case 3: We have *m* number of vertices with label 1 in $C_{(1,m-1)}$ and *n* number of vertices with label 2 in $C_{(1,n-1)}$

Note that, $e_f(1) = mn+m+1$ and $e_f(2) = n+1$. Then, $e_f(1)-e_f(2) = mn+m+1-1-n = mn > 1$ as $n = m$.

Case 4: We have *m* number of vertices with label 2 in $C_{(1,m-1)}$ and *n* number of vertices with label 1 in $C_{(1,n-1)}$

Note that, $e_f(1) = mn+n+1$ and $e_f(2) = m+1$. Then, $e_f(1)-e_f(2) = mn+n+1-m-1 = mn > 1$ as $n = m$. Hence, $C_{(1,m-1)} \vee C_{(1,n-1)}$ is not HMC, where $m = n$ and $m \geq 2$.

Proposition 2.5.

$C_{(1,m-1)} \vee C_{(1,n-1)}$ is not HMC, where $m + n$ is even and $m,n \geq 2$.

Proof:

Note that, $|V(C_{(1,m-1)} \vee C_{(1,n-1)})| = n + m$. Suppose that $C_{(1,m-1)} \vee C_{(1,n-1)}$ is HMC. Since we have $|v_f(1)| = \frac{n+m}{2} = |v_f(2)|$.

Case 1: All the vertices of label 1 and 2 are in sequence in $C_{(1,n-1)}$ and $C_{(1,m-1)}$

Suppose that we have *r* number of vertices with label 1 in $C_{(1,n-1)}$. So, we have $(n - r)$ vertices of of label 2 in $C_{(1,n-1)}$. Hence, we have $\frac{n+m}{2} - r$ vertices of label 1 in $C_{(1,m-1)}$ and $m - \frac{n+m}{2} + r$ vertices of label 2 in $C_{(1,m-1)}$. Note that, $e_f(1) = (r+1) + (\frac{n+m}{2} - r + 1) + rm + (\frac{n+m}{2} - r)(n-r) + 1$ and $e_f(2) = (n - r - 1) + (m - \frac{n+m}{2} + r - 1) + (n - r)(m - \frac{n+m}{2} + r) + 1$. Then, $e_f(1)-e_f(2) = mr + 4 + n^2 - 3nr + 2r^2$. We know that $r = \frac{m+n}{2}$, So, we have $e_f(1) - e_f(2) > 1$.

Case 2: Some of the vertices of label 2 are not in sequence in $C_{(1,n-1)}$ and $C_{(1,n-1)}$

Suppose that we have *r* number of vertices with label 1 in $C_{(1,n-1)}$. So, we have $\frac{n+m}{2} - r$ vertices of label 1 in $C_{(1,m-1)}$. Hence, we have $(n - r)$ vertices of label 2 in $C_{(1,n-1)}$ and $(m - \frac{n+m}{2} + r)$ vertices of label 2 in $C_{(1,m-1)}$. Suppose that there exist *i* number of vertices with label 2 are not in sequence in $C_{(1,n-1)}$ and *j* number of vertices with label 2 are not in sequence in $C_{(1,m-1)}$. Note that, $e_f(1) = (r + i + 1) + (\frac{n+m}{2} - r + j + 1) + rm + (n - r)(\frac{n+m}{2} - r) + 2$ and $e_f(2) = (n - r - i - 1) + (m - \frac{n+m}{2} + r - j - 1) + (n - r)(m - \frac{n+m}{2} + r)$. Now, $e_f(2)$ in case 2 ≤ $e_f(2)$ in case 1 and $e_f(1)$ in case 2 ≥ $e_f(1)$ in case 1. So, $e_f(1) - e_f(2)$ in this case is ≥ $e_f(1) - e_f(2)$ in case 1. Now, we have already proved in case 1 that $e_f(1) - e_f(2) > 1$. Hence, in this case $e_f(1) - e_f(2) > 1$.

Case 3: *m* > *n*

Subcase 3.1: All the vertices in $C_{(1,n-1)}$ are with label 1

So, we have *n* number of vertices with label 1 in $C_{(1,n-1)}$. Suppose that we have *r* number of vertices with label 1 in $C_{(1,m-1)}$. So, there exist $m - r$ number of vertices with label 2 in $C_{(1,m-1)}$.

Subsubcase 3.1.1:All the vertices in $C_{(1,m-1)}$ are in sequence

Then, $e_f(1) = (n+1)+(r+1)+mn$ and $e_f(2) = m-r$. Then, $e_f(1)-e_f(2) = mn-m+n+2 > 1$ as $mn > m,n$.

Subsubcase 3.1.2: All the vertices with label 2 are not in sequence in $C_{(1,m-1)}$

Suppose that we have *i* number of vertices from $(m - r)$ number of vertices are not in sequence in $C_{(1,m-1)}$. Then, $e_f(1) = n + (r + i + 1) + mn + 2$ and $e_f(2) = m - r - i - 1$. Now, $e_f(2)$ in subsubcase 3.1.2 ≤ $e_f(2)$ in subsubcase 3.1.1 and $e_f(1)$ in subsubcase 3.1.2 ≥ $e_f(1)$ in subsubcase 3.1.1. So, $e_f(1) - e_f(2)$ in this case ≥ $e_f(1) - e_f(2)$ in subsubcase 3.1.1. Now, we have already proved in subsubcase 3.1.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this case.

Subcase 3.2: All the vertices in $C_{(1,n-1)}$ are with label 2

So, we have *n* number of vertices with label 2 in $C_{(1,n-1)}$. Suppose that we have *r* number of vertices with label 1 in $C_{(1,m-1)}$. So, there exist $m - r$ number of vertices with label 2 in $C_{(1,m-1)}$.

Subsubcase 3.2.1: All the vertices in $C_{(1,n-1)}$ are in sequence

Then, $e_f(1) = r + 2 + rn$ and $e_f(2) = n + m - r$. Then, $e_f(1) - e_f(2) = rn + 2r + 2 - n - m$.

We know that $r = \frac{n+m}{2}$. So, $e_f(1) - e_f(2) > 1$.

Subsubcase 3.2.2: All the vertices in $C_{(1,n-1)}$ are not in sequence

Suppose that we have *i* number of vertices from $(n - r)$ number of vertices are not in sequence in $C_{(1,m-1)}$. Then, $e_f(1) = r + i + 1 + rn + 2$ and $e_f(2) = m - r - i - 2 + n + n(m - r)$. Now, $e_f(2)$ in subsubcase 3.2.2 ≤ $e_f(2)$ in subsubcase 3.2.1 and $e_f(1)$ in subsubcase 3.2.2 ≥ $e_f(1)$ in subsubcase 3.2.1. so, $e_f(1)-e_f(2)$ in this case ≥ $e_f(1)-e_f(2)$ in subsubcase 3.2.1. Now, we have already proved in subsubcase 3.2.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this case.

Hence, $C_{(1,m-1)} \vee C_{(1,n-1)}$ is not HMC, where $n + m$ is even and $m,n \geq 2$.

Proposition 2.6.

$C_{(1,m-1)} \vee C_{(1,n-1)}$ is not HMC, where $m + n$ is odd and $m,n \geq 2$.

Proof:

Note that, $|V(C_{(1,m-1)} \vee C_{(1,n-1)})| = n+m = 2k+1$ where $k \in \mathbb{N}$. Suppose that $C_{(1,m-1)} \vee C_{(1,n-1)}$ is HMC. Without loss of generality, we may assume that $m > n$.

In this case we have two possibilities. $(i) v_f(1) = \frac{m+n+1}{2}$ and $v_f(2) = \frac{m+n-1}{2}$ $(ii) v_f(1) = \frac{m+n-1}{2}$ and $v_f(2) = \frac{m+n+1}{2}$

So, we consider the following cases.

Case 1: $v_f(1) = \frac{n+m+1}{2} = k+1$ **and** $v_f(2) = \frac{n+m-1}{2} = k$

Subcase 1.1: All the vertices of label 1 are in sequence in $C_{(1,n-1)}$ and $C_{(1,m-1)}$

Then, it is clear that all the vertices of label 2 are in sequence in $C_{(1,n-1)}$ and $C_{(1,m-1)}$. Suppose that we have r number of vertices with label 1 in $C_{(1,n-1)}$. So, we have $(n - r)$ vertices of of label 2 in $C_{(1,n-1)}$. Hence, we have $(k + 1 - r)$ vertices of label 1 in $C_{(1,m-1)}$ and $(k - n + r)$ vertices of label 2 in $C_{(1,m-1)}$. Note that, $e_f(1) = (r + 1) + (k + 2 - r) + rm + (k + 1 - r)(n - r) + 1$ and $e_f(2) = (n - r - 1) + (k - n + r - 1) + (n - r)(k - n + r) + 1$. Then, $e_f(1) - e_f(2) = (n - r)^2 + 5 + rm + (n - r)(1 - r) = (n - r)(n + 1 - 2r) + rm + 5$. Now, $e_f(1) - e_f(2) > 1$ if $n + 1 \geq 2r$. If $n + 1 < 2r$, then $\frac{(n+1)}{2} < r$. Now, $r + k = \frac{m+n+1}{2} > \frac{(n+r)}{2} + k$. Therefore, $m > k$. Suppose that $r = \frac{(n+1)}{2} + l$. Then, $e_f(1) - e_f(2) = 2l^2 + 2l + \frac{1}{2} + lm + 5 > 1$.

Subcase 1.2: Some of the vertices of label 2 are not in sequence in $C_{(1,n-1)}$ and $C_{(1,m-1)}$

Suppose that we have r number of vertices with label 1 in $C_{(1,n-1)}$. So, we have $(n - r)$ vertices of label 2 in $C_{(1,n-1)}$. Hence, we have $(k - r)$ vertices of label 1 in $C_{(1,m-1)}$ and $(k + 1 - n + r)$ vertices of label 2 in $C_{(1,m-1)}$. Suppose that there exist l number of vertices with label 2 are not in sequence in $C_{(1,n-1)}$ and j number of vertices with label 2 are not in sequence in $C_{(1,m-1)}$. Note that, $e_f(1) = (r + l + 1) + (k - r + j + 2) + rm + (n - r)(k + 1 - r) + 2$ and $e_f(2) = (n - r - l - 1) + (k - n + r - j - 1) + (n - r)(k - n + r)$. Now, $e_f(2)$ in subcase 1.2 $\leq e_f(2)$ in subcase 1.1 and $e_f(1)$ in subcase 1.2 $\geq e_f(1)$ in subcase 1.1. So, $e_f(1) - e_f(2)$ in this case is $\geq e_f(1) - e_f(2)$ in subcase 1.1. Now, we have already proved in subcase 1.1 that $e_f(1) - e_f(2) > 1$.

Hence, $e_f(1) - e_f(2) > 1$ in this case .

Case 2: $v_f(1) = \frac{n+m-1}{2} = k$ **and** $v_f(2) = \frac{n+m+1}{2} = k+1$

Subcase 2.1: All the vertices of label 1 are in sequence in $C_{(1,n-1)}$ and $C_{(1,m-1)}$

Then, it is clear that all the vertices of label 2 are in sequence in $C_{(1,n-1)}$ and $C_{(1,m-1)}$. Suppose that we have r number of vertices with label 1 in $C_{(1,n-1)}$. So, we have $(n - r)$ vertices of label 2 in $C_{(1,n-1)}$. Hence, we have $(k + 1 - r)$ vertices of label 1 in $C_{(1,m-1)}$ and $(k - n + r)$ vertices of label 2 in $C_{(1,m-1)}$. Note that, $e_f(1) = (r + 1) + (k + 1 - r) + rm + (k - r)(n - r) + 1$ and $e_f(2) = (n - r - 1) + (k - n + r) + (n - r)(k - n + r + 1) + 1$. Then, $e_f(1) - e_f(2) = (n - r)^2 + 3 + rm + (r - n)(1 + r) = (n - r)(n - 1 - 2r) + rm + 3$. Now, $e_f(1) - e_f(2) > 1$

if $n \geq 1 + 2r$. If $n < 1 + 2r$, then $\frac{(n-1)}{2} < r$. Now, $r + k = \frac{m+n-1}{2} > \frac{(n-r)}{2} + k$. Therefore, $m > k$. Suppose that $r = \frac{(n-1)}{2} + l$. Then, $e_f(1) - e_f(2) = (\frac{mn}{2} - \frac{m}{2}) + 3 + l(m - n - 1) > 1$, if $m \geq n + 1$. Suppose that $m \leq n + 1$. Then since, $m \geq n$, we have $m = n + 1$. So, we have

$e_f(1) - e_f(2) = (\frac{mn}{2} - \frac{m}{2}) + 3 + l(m - n - 1) = (\frac{mn}{2} - \frac{m}{2}) + 3 > 1$.

Subcase 2.2: Some of the vertices of label 2 are not in sequence in $C_{(1,n-1)}$ and $C_{(1,m-1)}$

Suppose that we have r number of vertices with label 1 in $C_{(1,n-1)}$. So, we have $(n - r)$ vertices of label 2 in $C_{(1,n-1)}$. Hence, we have $(k - r)$ vertices of label 1 in $C_{(1,m-1)}$ and $(k + 1 - n + r)$ vertices of label 2 in $C_{(1,m-1)}$. Suppose that there exist l number of vertices with label 2 are not in sequence in $C_{(1,n-1)}$ and j number of vertices with label 2 are not in sequence in $C_{(1,m-1)}$.

Note that, $e_f(1) = (r + l + 1) + (k - r + j + 1) + rm + (n - r)(k - r) + 2$ and $e_f(2) = (n - r - l - 1) + (k - n + r - j) + (n - r)(k - n + r + 1)$. Now, $e_f(2)$ in subcase 2.2 $\leq e_f(2)$ in subcase 2.1 and $e_f(1)$ in subcase 2.2 $\geq e_f(1)$ in subcase 2.1. So, $e_f(1) - e_f(2)$ in this case is $\geq e_f(1) - e_f(2)$ in subcase 2.1. Now, we have already proved in subcase 2.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this case.

Case 3: $m > n$

Subcase 3.1: All the vertices in $C_{(1,n-1)}$ are with label 1

So, we have n number of vertices with label 1 in $C_{(1,n-1)}$. Suppose that we have r number of vertices with label 1 in $C_{(1,m-1)}$. So, there exist $m - r$ number of vertices with label 2 in $C_{(1,m-1)}$.

Subsubcase 3.1.1: All the vertices in $C_{(1,m-1)}$ are in sequence

Then, $e_f(1) = n + (r + 1) + mn + rn + 1$ and $e_f(2) = m - r$. Then, $e_f(1) - e_f(2) = (mn - m) + n + 2r + rn + 2 > 1$ as $mn > m$.

Subsubcase 3.1.2: All the vertices with label 2 are not in sequence in $C_{(1,m-1)}$

Suppose that we have l number of vertices from $(m - r)$ number of vertices are not in sequence in $C_{(1,m-1)}$. Then, $e_f(1) = n + (r + l + 1) + mn + 2$ and $e_f(2) = m - r - l - 1$. Now, $e_f(2)$ in subsubcase 3.1.2 $\leq e_f(2)$ in subsubcase 3.1.1 and $e_f(1)$ in subsubcase 3.1.2 ≥ $e_f(1)$ in subsubcase 3.1.1. So, $e_f(1)-e_f(2)$ in this case is ≥ $e_f(1)-e_f(2)$ in subsubcase 3.1.1. Now, we have already proved in subsubcase 3.1.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this case.

Subcase 3.2: All the vertices in $C_{(1,n-1)}$ are with label 2

So, we have n number of vertices with label 2 in $C_{(1,n-1)}$. Suppose that we have r number of vertices with label 1 in $C_{(1,m-1)}$. So, there exist $m - r$ number of vertices with label 2 in $C_{(1,m-1)}$.

Subsubcase 3.2.1: All the vertices in $C_{(1,n-1)}$ are in sequence

Then, $e_f(1) = r + 2 + rn$ and $e_f(2) = n + m - r$. Then, $e_f(1) - e_f(2) = nr - n - m + 2r + 2$.

In this case we have two possibilities.

(i) Suppose that $r = \frac{n+m+1}{2}$. So, $e_f(1) - e_f(2) = \frac{mn}{2} + \frac{n^2}{2} + \frac{n}{2} + 3 > 1$

(ii) Suppose that $r = \frac{n+m-1}{2}$. So, $e_f(1) - e_f(2) = \frac{mn}{2} + (\frac{n^2}{2} - \frac{n}{2}) + 3 > 1$.

Subsubcase 3.2.2: All the vertices in $C_{(1,n-1)}$ are not in sequence

Suppose that we have l number of vertices from $(n - r)$ number of vertices are not in sequence in $C_{(1,m-1)}$. Then, $e_f(1) = r + l + 3 + rn$ and $e_f(2) = m - r - l - 1 + n + n(m - r)$. Now, $e_f(2)$ in subsubcase 3.2.2 ≤ $e_f(2)$ in subsubcase 3.2.1 and $e_f(1)$ in subsubcase 3.2.2 ≥ $e_f(1)$ in subsubcase 3.2.1. So, $e_f(1) - e_f(2)$ in this case is ≥ $e_f(1) - e_f(2)$ in subsubcase 3.2.1. Now, we have already proved in subsubcase 3.2.1 that $e_f(1) - e_f(2) > 1$. Hence, $e_f(1) - e_f(2) > 1$ in this case. Hence, $C_{(1,m-1)} \vee C_{(1,n-1)}$ is not HMC, where $n + m$ is odd and $m,n \geq 2$.

Theorem 2.2. $C_{(1,m-1)} \vee C_{(1,n-1)}$ is not HMC, where $n,m \in \mathbb{N}$, $m,n \geq 2$.

Proof:

Proof follows from propositions 2.4, 2.5 and 2.6.

3. CONCLUSION

In this article, we have discussed Harmonic mean cordial labeling of $K_n \vee C_{(1,\, m-1)}$ and $C_{(1,m-1)} \vee C_{(1,n-1)}$ for any $n,m \geq 2$ and $n,m \in \mathbb{N}$.

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