

A Discrete Time Geometric Queue Attached to an Inventory under a (0, Q) Policy

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Abstract: This paper describes how a threshold policy, or $TP = [(0, Q), 1]$ -based inventory management, economically offers customers a single product with a discrete time geometric queue attached to a Geo/Geo/1 service base. The order for Q units starts when inventory $I(t)$ at time 't' drops to the level '0', and the length of the queueing process $X(t)$ is at least one. We focus on the process $Z(t) = (X(t), I(t))$, $t \in [0, \infty)$. We then analyse the steady state probabilities and then provides a numerical study for graphing the economic order quantity.

Keywords: stochastic processes, threshold policy, stationary distribution, order quantity.

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1. INTRODUCTION

Stochastic processes and their applications help researchers to solve the queue of customers who need one or more products in a storage system. This article examines a queueing inventory system (QIS) that operates under a threshold order policy $TP = [(0, Q), 1]$. The traditional order policy (0, Q) with order level '0' used for an individual inventory system is modified by an addition of at least '1' waiting customer of a discrete-time Geo/Geo/1 queue for an order placement of 'Q' units

The inventory management takes best decisions by monitoring the on-hand inventory process $I(t) \in \{0, 1, \dots, Q\}$ jointly with the queue length process $X(t) \in \{0, 1, \dots\}$ of the discrete-time Geo/Geo/1 queueing system. The term QIS was first introduced by [31], [23] and [40]. In addition, QIS has many applications governing different activities of supply chain managements, see [28], [19] for retrial customer queues, and [2] for medical service. Other researchers [10], [14] and [22] studied impacts of seller-buyer integration in inventory problems.

Several methods have been used to find service descriptions for QIS. For example, [3, 4, 5, 6, 7, 8], investigated the behaviour of different types of inventory stocks tied to M/M/1 queues. Contributions have been made for different types of QIS either with lost sales or back-order customers in [9], [27, 29], [28] and [30]. [18] conducted a survey of inventory samples focused on active service times. [19] reviewed papers on queued inventory systems. [38] came up with a digital solution for a two-stage inventory and production system with random incoming demand. Authors of [20] developed a method to achieve optimal inventory management order quantities with stock-dependent replaceable products. [39] and [26] analysed M/M/1 queues with inventory over lost sales, and estimated overall delays.

Authors of [13] studied an economic order quantity (EOQ) model where new approach was developed to optimize the EOQ of the model. [9] investigated a QIS for a two-vendor system under (s, S) policy with active service times. [1]

discussed a typical type of QIS allowing feedback customers and obtained several stationary characteristics of the inventory system.

[34] considered an optimal inventory policy for monotonic purchase price variation where the price change depends on order quantity. [16] investigated a queue-dependent service rates in the stochastic queueing-inventory system (SQIS) under (s, S) inventory policy and obtained reduced form of the rate matrix as non-negative minimal solutions of the quadratic systems through the matrix-geometric method developed by [25]. [35] studied the inventory systems monitoring monotonic changes of prices changes of commodities.

[24] proposed an inventory model for evaluating the stationary distribution with a policy $(R, s, \ln Q)$ using approximation bounds. [42] investigated a QIS with server's vacations and impatient customers under continuous review (s, S) policy. [31] suggested approximation procedures to find performance descriptions for a typical type of QIS. Basic details on queueing models with impatient customers and quasi birth-death processes can be found in [32] and [33].

1.1 Discrete Time QIS under $TP = [(0, Q), 1]$ order policy

Inventory policy "TP": The underlying service function of the proposed QIS is Discrete Time Queuing (DTQ). The inventory policy of the present study is of the threshold type, $TP = [(0, Q), 1]$ which operates subject to two conditions, (i) inventory level is zero and (ii) simultaneously queue length ≥ 1 , as the requirement for placing an order of Q units.

A primary guideline of the policy 'TP' warrants at time 't' that an order is triggered when the inventory level $I(t)$ is at '0' and there is at least '1' (i.e., $X = X(t) \geq 1$) customers in the system. There can be at most one pending order, with $[(0, Q), 1]$ being the order point, and the order quantity is a bulk of Q units for each replenishment.

The State space E_{TP} : We observe that the underlying sequence $Z(t) = (X(t), I(t))$ is associated with the queueing-inventory system (QIS) described above and it forms a Markov process on the state space:

$$E_{TP} = N_0 \times M_0 \quad (1)$$

$$N_0 = \{0, 1, \dots, \infty\} \text{ and } M_0 = \{0, 1, \dots, Q\} \quad (2)$$

However, the Z process does not reach the state $(0, Q)$ because the replenishment order must not be triggered if the server sees the system on the state $(0, 0)$.

Back-orders and Lost-Sales: Whenever the server is idle to serve customers and inventory level is zero, this service will start when the next replenishment arrives. These customers (those who arrive during this replenishment period while stock is not available) are added to the back-order list. If an incoming request perceives an empty queue length, then it immediately goes to the server or joins the queue and thus the system is a QIS with back order, say QISB

We now introduce a modified version of QISB called Queuing Inventory System with Lost (QISL) sales and a reflection Barrier State $(0,0)$ as an alternative example.

For the proposed model QISB, time is assumed to be slotted with the following assumptions:

1. The service time axis is decomposed into an unlimited number of slots, each with unit length.
2. All QIS's activities of arriving time, lead time and departing time happen at the slot boundaries
3. A potential arrival and a placement of order occur in the interval $(m-, m)$ and either a potential departure or a replenishment of an order can take place in the interval $(m, m+)$.
4. The arrival process is a Bernoulli process with probability of success 'p' and hence the number of inter time intervals of the arrivals follow the geometric distribution with the same rate 'p'.
5. The replenishment process is Bernoulli process with probability of success 'r' and hence the inter time intervals of the lead times follow the same geometric distribution with rate 'r'.
6. The service times are independent and identically distributed according to a geometric distribution with rate 'q'.
7. The arrival process, replenishment process and the service process are all mutually independent

That is, time axis is divided into unlimited number of fixed length intervals known as slots. Here, customers arrive according to a Bernoulli process with success probability p . Each customer, who is served according to a geometrically distributed service time with success probability q , takes exactly one item from the inventory. Consequently, the demand rate of the inventory is equal to p if the state of the system is at $(0, 0)$; otherwise, the demand rate is equal to the service rate q .

The random replenishment lead time, which is the time span between ordering of materials and receipt of the goods, is geometrically distributed with success probability r . The entire order is received into stock at the same time. This type of inventory system is defined to be a discrete review system where the inventory state is inspected during slot boundaries and orders are placed every time the inventory on hand reaches the reorder point 0 and simultaneously the queue length value is strictly positive.

The main objective of this investigation is related with the problem of how the classical performance measures (e.g., queue length, waiting time, etc.) could help the management of the attached inventory to operate the $TP = [(0, Q), 1]$ policy in an optimal way and how inventory management must minimize the queue length of demands or the average sojourn times of demands to directly enable cost optimization in this integrated model.

For mathematical clarity, we assume that the departures or the replenishment occur in the interval $(m, m+)$, and the arrivals and the placement of orders occur in the interval $(m-, m)$; $m-$ is the moment immediately before the slot boundaries and $m+$ is the moment immediately after the slot boundaries.

Researchers have been applying discrete-time queues (DTQs) in a number of slotted digital communication and queueing systems, see in [15]. The terminologies of these DTQs scattering in [11], [36], [37], and [12] have been used in the proposed QISB and QISL cases.

Section 2 provides necessary introduction to the model the QISB frame work. The quasi birth and death (QBD) process representing the queue length and the inventory level of the QISB is described and the stationary distribution of the QBD is obtained by matrix analysis. Section 3 focusses on the stationary performance characteristics and the formulation of a total expected cost (TEC) function in terms of holding cost, ordering and waiting time costs. For a given data set, a graph showing the relationship between TEC and the order quantity Q is drawn to locate and read the EOQ. Section 4 applies the methodology used in the sections 2 and 3 to analyze the QISL case. In addition, using the same data set (already considered in in the section 3) TEC values for different Q values in the range 2 to 12 are calculated and thus the EOQ is also found out. A formal conclusion report is given in the section 5.

2. QISB WITH AN UNDERLYING GEO/GEO/1 QUEUE

Assume that the three queueing activities happen according to arriving (arrival first), departing (service completion next) and replenishing (order delivery) at the slot boundaries. A potential arrival occurs in the interval $(m-, m)$, and a potential departure and either triggering of an order or the replenishment of the order (that was already placed) take place in the interval $(m, m+)$.

Let the stationary state process be denoted by $Z = (X, I) = \lim_{t \rightarrow \infty} \{Z(t) = (X(t), I(t))\}$. Applying standard probabilistic arguments on the limiting bivariate process Z , it can be shown that the process Z is regular and irreducible over the truncated space $[E_{TP} - \{(0, Q)\}]$. In this model, an arriving customer begins its service instantly if it finds the server idle. Otherwise, i.e., if the server is busy, the arrival joins the waiting line at the back end of the queue. The stationary distribution of the queue length of the system is obtained. Further, the distribution of sojourn time of a customer is also computed.

2.1 Generator matrix of the QBD process Z

Since the Z process is a level-dependent QBD process, using the properties of the QBD under the TP-policy, we organize the elements of the transition probability matrix (TPM) in (3), say G :

$$G = (g(L_i, L_j)) \tag{ 3}$$

$$L_i = (((i, 0), (i, 1), \dots, (i, Q)); i \in N_0 \text{ and } \cup_{i=0}^{\infty} L_i = E_{TP}) \tag{ 4}$$

where L_i of (4) is called the i^{th} level vector of size or order $(Q+1)$. The tridiagonal form of the TPM is given in (5):

$$G = \begin{pmatrix} B_1 & A_0 & \mathbf{0} & \mathbf{0} & \dots & \dots \\ A_2 & A_1 & A_0 & \mathbf{0} & \dots & \dots \\ \mathbf{0} & A_2 & A_1 & A_0 & \dots & \dots \\ \mathbf{0} & \mathbf{0} & A_2 & A_1 & \ddots & \vdots \\ \vdots & \vdots & \dots & \ddots & \ddots & \ddots \\ \dots & \dots & \dots & \dots & \ddots & \ddots \end{pmatrix} \tag{5}$$

Each sub-matrix of G of (5) is a square matrix of order (Q + 1) indexed by j=0, 1, . . . , Q. The precise structures of B1, A0, A1, and A2 are described in (6):

$$B_1 = \begin{pmatrix} \bar{p} & 0 & \dots & 0 & 0 \\ pq & \bar{p} & \dots & 0 & 0 \\ 0 & pq & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & pq & \bar{p} \end{pmatrix} \quad A_0 = \begin{pmatrix} p\bar{r} & 0 & \dots & 0 & pr \\ 0 & p\bar{q} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & p\bar{q} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \bar{p}q & 0 & \dots & 0 & 0 \\ 0 & \bar{p}q & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{p}q & 0 \end{pmatrix} \quad A_1 = \begin{pmatrix} \bar{p}\bar{r} & 0 & \dots & 0 & \bar{p}r \\ pq & \bar{p}\bar{q} & \dots & 0 & 0 \\ 0 & pq & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & pq & \bar{p}\bar{q} \end{pmatrix} \tag{6}$$

Therefore, inventory level process I is Markov process with an underlying TPM ‘A’ over the sates {0, 1, . . . , Q} of order (Q+1).

$$A = A_0 + A_1 + A_2 = \begin{pmatrix} \bar{r} & 0 & \dots & 0 & r \\ q & \bar{q} & \dots & 0 & 0 \\ 0 & q & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & q & \bar{q} \end{pmatrix} \tag{7}$$

We claim that the matrix A of (7) serves as a TPM of a finite, irreducible, discrete time Markov process with stationary distribution $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_Q)$, where $\alpha_0 + \alpha_1 + \dots + \alpha_Q = 1$. Denote the unit vector of size (Q+1) by $e = (1, 1, \dots, 1)$. Then,

$$\alpha A = \alpha, \quad \alpha e = 1 \tag{8}$$

Solving (8), one can find the unique solution for α . Thus, it is given by (9),

$$\alpha_0 = \frac{q}{q + Q\gamma}, \text{ and } \alpha_j = \frac{\gamma}{q + Q\gamma} \text{ for } j = 1, \dots, Q \tag{9}$$

Matrix analysis solutions for several level-independent versioning types of quasi-birth-death (QBD) processes have been extensively studied by many researchers and are dealt with lucidly in excellent texts of [21] and [25]. Such a solution for the case Two Server Poisson Queues with a Slow Service Provider for Impatient Customers is investigated in [32] and for a Quasi birth-death processes of two-server queues with stalling in [33].

Distribution: Denote by $\Pi = (\pi_0, \pi_1, \pi_2, \dots)$, the steady state distribution of Z and by π_n the row vector $\pi_n = (\pi(n, 0), \dots, \pi(n, Q))$ associated to level $L_n, n \in N_0$. Let $e = (1, 1, \dots)$ be a column vector of unit elements. As the state (0, Q) cannot be accessible from any other state (0, j) for j=0,1,..., (Q-1), we must set $\pi(0, Q) = 0$.

Lemma 1: If the inter-level generator matrix A is irreducible, then the sequence Z of states on the state space E_{TP} defined by (1) and (2) forms a positive recurrent process if and only if

$$\rho = \frac{\alpha A_0 e}{\alpha A_2 e} = \frac{p}{\bar{p}q} \left(\frac{\bar{q} + q\alpha_0}{1 - \alpha_0} \right) < 1 \tag{10}$$

2.2 Steady state characteristics of the inventory cum queueing management

We now find an optimal policy for the inventory system with the underlying Z process that has the TPM G of (5) subject to the stability condition of (10). In [4], [6] and [5], they have shown that the ordering policy ($r \geq 0, Q$) is only 3442

sub optimal. They have further proved that optimal policies play crucial role as decision variable in the queue length calculation. Such of those optimal policies are of monotone structure. Let

$$K_0 = \alpha A_2 (I - B_1)^{-1} \text{ and } K = K_0 e \tag{11}$$

Theorem 1: For some positive constant $\beta > 0$, and R = a diagonal matrix of order $(Q+1)$, and substitution of $n= 1, 2, \dots$, in (12)

$$\pi_n = \beta \alpha R^n \tag{12}$$

leads to

$$R = \text{diag}(\rho), \quad \pi_0 = \beta \rho K_0 \tag{13}$$

$$\beta = \frac{1 - \rho}{(\bar{q} + q \alpha_0)(1 - \rho) + \rho} \tag{14}$$

Proof: Solving the matrix equations $\Pi G = \Pi e = 1$ we obtain the following set of matrix equations in terms of the sub-matrices of G and sub-vectors of Π :

$$\pi_0 B_1 + \pi_1 A_2 = \pi_0 \tag{15}$$

$$\pi_{n-1} A_0 + \pi_n A_1 + \pi_{n+1} A_2 = \pi_n \text{ for } n=2, 3, \dots \tag{16}$$

Substituting the given $\pi_n = \beta \alpha R^n$ of (12) for $n=2, 3, \dots$, in(16), we see that

$$\beta \alpha R^{n-1} (A_0 + R A_1 + R^2 A_2 = R) \tag{17}$$

Statement (17) indicates that explicit values to all members of the R matrix can be found by solving the equation (18):

$$A_0 + R A_1 + R^2 A_2 = R \tag{18}$$

Add $R A_2$ on both sides of (18), we get,

$$\begin{aligned} A_0 + R [A_1 + (I + R) A_2] &= R + R A_2 \rightarrow A_0 + R\{(I - R)^{-1} [(I - R)A_1 + (I - R)(I + R) A_2]\} = R + R A_2 \\ \rightarrow A_0 + R\{(I - R)^{-1} [(A_1 + A_2) - (R A_1 + R^2 A_2)]\} &= R + R A_2 \\ \rightarrow A_0 e + R\{(I - R)^{-1} [(A_0 + A_1 + A_2) - (A_0 + R A_1 + R^2 A_2)]\} e &= R e + R A_2 e \\ \rightarrow A_0 e + R\{(I - R)^{-1} [I - R]\} e &= R e + R A_2 e \\ \rightarrow A_0 e + R e &= R e + R A_2 e \\ \rightarrow A_0 e = R A_2 e \rightarrow \alpha A_0 e = \alpha R A_2 e \end{aligned} \tag{19}$$

The value $\rho = (\alpha A_0 e)/(\alpha A_2 e)$ stated by (10) of the (Lemma 1), can also be obtained from (19) if and only if $R = \text{diag}(\rho)$. Using this fact, we now claim that

$$R = \text{diag}(\rho) \text{ and } \pi_n = \beta \rho^n \alpha \text{ for } n=1, 2, \dots \tag{20}$$

Arranging the equation stated by (15) the value of π_1 from (12), we simplify that

$$\pi_0 = \pi_1 A_2 (I - B_1)^{-1} \tag{21}$$

$$\rightarrow \pi_0 = \beta \rho \alpha [A_2 (I - B_1)^{-1}] \tag{22}$$

Using (22) we can compute $\pi_0 e$. Using (11), we have

$$K = \alpha A_2 (I - B_1)^{-1} e \tag{23}$$

Now using (23), the equation (22) can be rewritten:

$$\pi_0 e = K \beta \rho \tag{24}$$

Substituting the $\{\pi_n e \text{ for } n= 0, 1, \dots, \infty\}$ from (24) and (12) into the normalizing condition $\sum_{n=0}^{\infty} \pi_n e = 1$ we can calculate the value of the proportionality constant β . Thus,

$$\beta = \frac{1 - \rho}{(\bar{q} + q \alpha_0)(1 - \rho) + \rho} \tag{25}$$

Thus, all the statements of the Theorem 1 are established.

Lemma 2: The vector K is calculated by

$$K = \begin{pmatrix} \frac{\bar{p} q}{p} (\alpha_1 - q\alpha_2 + q^2 \alpha_3 - \dots + (-q)^{(Q-1)} \alpha_Q, \\ \alpha_2 - q\alpha_3 + q^2 \alpha_4 - \dots + (-q)^{(Q-1)} \alpha_Q, \\ \dots, \dots, \dots, \\ (-q)^2 \alpha_{(Q-1)} + (-q) \alpha_Q, \\ \alpha_Q, 0 \end{pmatrix}$$

Proof: Notice that the matrix $(I - B_1)^{-1}$ and its inverse matrix are Lower-Triangular Matrices and the matrix A2 is Strictly Lower-Triangular Matrix. Hence the product $A_2 (I - B_1)^{-1}$ would be a Lower-Triangular Matrix.

$$A_2(I - B_1)^{-1} = \left(\frac{\bar{p}q}{p} \right) \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ -q & 1 & \ddots & 0 & 0 & 0 \\ q^2 & -q & \ddots & \ddots & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ (-q^{(Q-1)}) & (-q^{(Q-2)}) & \dots & -q & 1 & 0 \end{pmatrix} \tag{26}$$

Thus the Lemma 2 is established and we see that $\pi(0, Q)=0$ as expected from (26).

3. PERFORMANCE MEASURES

In this section, we explore briefly how the above integration of queueing-inventory system produce new results. Denote the conditional average inventory levels: $\bar{I}(0)$ given $X=0$, and \bar{I} given $X>0$. Using the results of the preceding inter-play and the stable distributions, we get, using:

$$\bar{I}(0) = \sum_{j=0}^{Q-1} j \pi(0, j) = \text{and } \bar{I} = \sum_{n=1}^{\infty} \sum_{j=0}^Q j \pi(n, j) \tag{27}$$

The marginal mean queue length in the system (queue + service) L and the queue length (excluding the service) L_q are given by

$$L = \sum_{n=0}^{\infty} n \pi_n \text{ e and } L_q = \sum_{n=0}^{\infty} (n - 1) \pi_n \text{ e} \tag{28}$$

The average waiting time \bar{W} in the system and average time spent \bar{W}_q in the waiting line (excluding the service time) for a customer demand can now be computed using the ‘Little’s law’ of the queueing theory. Thus

$$\bar{W} = L / (p\bar{q}) \text{ and } \bar{W}_q = L_q / (p\bar{q}) \tag{29}$$

It establishes a relationship between the average number of customers in the system, the mean arrival rate and the mean customer response time (time between entering and leaving the system after getting service) in the steady state.

$$MRR = q \sum_{n=1}^{\infty} \pi(n + 1, 1) \tag{30}$$

3.1 Numerical Illustration

Figure 1: TEC vs Q for QISB with Back order

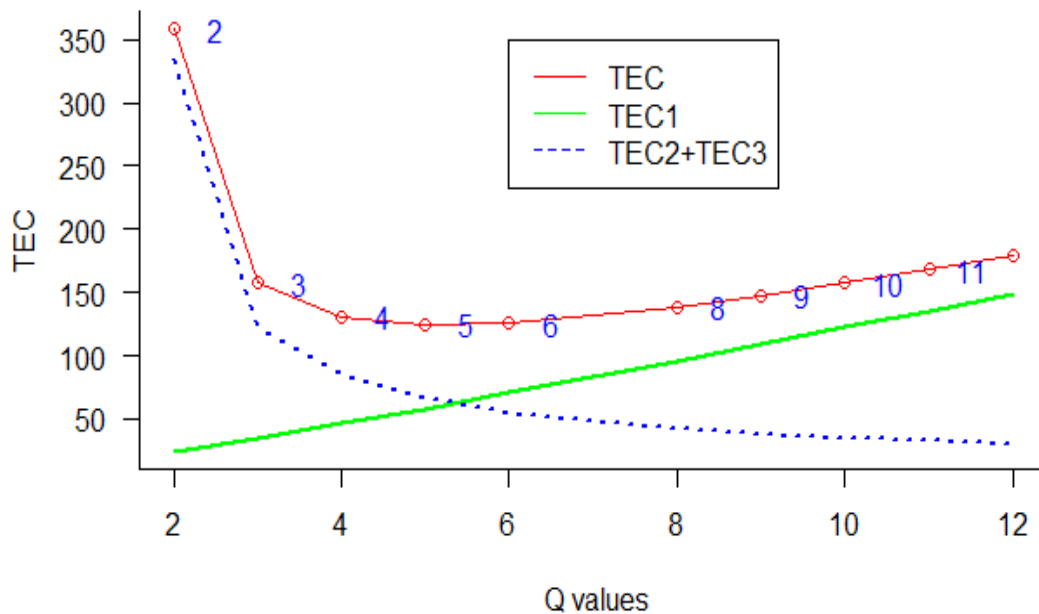


Figure 1: Q values vs TEC, $p=0.3$, $q=0.55$, $r=0.42$ and holding, ordering, and waiting costs ($h=27.5$; $A=84$; $V_w=12$).

We now illustrate a numerical exercise for tracing an Economic Order quantity (EOQ), say Q^* , that minimizes a linear total expected cost function $TEC(Q) = TEC(p, q, r, h, A, V_w, Q)$ stated in (31):

$$TEC(Q) = \bar{I} h + MRR(A) + \bar{W}_q V_w \tag{31}$$

For this case, input values are randomly selected $p=0.32$, $q=0.55$, $r=0.42$, $h=27.5$, $A=825.5$, $V_w=2.5$ such that these values satisfy the steady state condition of the Z-process under study for $Q = \{2, 4, 5, 6, 8, 9, 10, 11, 12, 13\}$.

The corresponding values of the TEC are computed, and the resulting Inventory cost vs (MRR cost + Wait cost) and TEC are drawn in Figure 1. The minimum TEC ($Q^* = 5$) = 124.5, when $Q=5$.

4. DISCRETE TIME QIS WITH A CONDITIONAL LOST CUSTOMER DEMAND

We now study an Inventory Queuing System with Lost Sales i.e., QISL. We assume that state (0, 0) is a reflective barrier to state (1,0). For the purposes of analyzing the new QISL example, we continue to retain all other QISB conditions. Again, there is at most one pending order and the state (0, Q) does not communicate with any other state in the QISL. These two QISB and QISL are new models in the field of SCM. It is not necessary to notice that the special reflection barrier state (0,0) can be accessed only from state (1,1) when the customer leaves.

We now introduce a modified version of QISB called as a queueing inventory system with lost sales (QISL) and a reflection barrier state (0,0) as an alternative example. We assume that state (0, 0) is a reflective barrier to state (1,0). For the purpose of analyzing the new QISL example, we continue to retain all other QISB conditions. Once more, there is at most a single unfulfilled order pending and the state (0, Q) does not communicate with any other state of the QISL. These two QISB and QISL are new models in the field of supply chain managements. This special reflecting barrier state (0,0) can be reached from the state (1,1) by a departure of customer.

Whenever the server is idle to serve customers and the stock is zero, the service will start when the next replenishment arrives. Customers who arrive during this inventory period when stock is unavailable will be discarded. Therefore, an incoming request is considered a lost customer if it is discovered that the stock is empty and is not allowed to join the queue.

Each sub-matrix of the infinitesimal generator, say G , of the QISL is a square matrix of order $(Q + 1)$ indexed by $j=0, 1, \dots, Q$. The precise structures of B_1, A_0, A_1 , and A_2 are described below:

$$\begin{aligned}
 B_1 &= \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ pq & \bar{p} & \dots & 0 & 0 & 0 \\ 0 & pq & \ddots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & pq & \bar{p} \end{pmatrix} & B_0 &= \begin{pmatrix} \bar{r} & 0 & \dots & 0 & 0 & r \\ 0 & p\bar{q} & \dots & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 & p\bar{q} \end{pmatrix} \\
 A_2 &= \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ \bar{p}q & 0 & \dots & 0 & 0 & 0 \\ 0 & \bar{p}q & \ddots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \bar{p}q & 0 \end{pmatrix} & A_1 &= \begin{pmatrix} \bar{r} & 0 & \dots & 0 & 0 & r \\ pq & \bar{p}\bar{q} & \dots & 0 & 0 & 0 \\ 0 & pq & \ddots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \dots & \ddots & pq & \ddots & \vdots \\ 0 & 0 & \dots & 0 & pq & \bar{p}\bar{q} \end{pmatrix} \tag{32}
 \end{aligned}$$

$$A_0 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & p\bar{q} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & p\bar{q} \end{pmatrix}$$

Solving the matrix equations $\Pi G = \Pi, \Pi e = 1$ we obtain the following set of matrix equations in terms of the sub-matrices of G and sub-vectors of Π , where $\Pi = (\pi_0, \pi_1, \pi_2, \dots)$.

$$\begin{aligned}
 \pi_0 B_1 + \pi_1 A_2 &= \pi_0 \\
 \pi_0 B_0 + \pi_1 A_1 + \pi_2 A_2 &= \pi_0 \\
 \pi_{n-1} A_0 + \pi_n A_1 + \pi_{n+1} A_2 &= \pi_n \text{ for } n = 2, 3, 4, \dots \tag{33}
 \end{aligned}$$

Let us try, as before with $\pi_n = \beta \alpha R^n$ for $n=0, 2, \dots$ in (33). We see that

$$A_0 + R A_1 + R^2 A_2 = R \tag{34}$$

Let us choose the 'p, q, r' values such that $\rho = \frac{\alpha A_0 e}{\alpha A_2 e} < 1$ Subject to this condition, as before, we can show that

1. The rate matrix $R = \text{diag}(\rho)$, diagonal matrix of order $(Q+1)$.
2. $\pi_0 = \pi_1 A_2 (I - B_1)^{-1} \rightarrow \pi_0 = \beta \rho \alpha [A_2 (I - B_1)^{-1}]$

Remark: Equation (middle) in (33) can be used to check if the correct solution vector Π satisfying the system $\Pi G = \Pi, \Pi e = 1$ exists or not.

4.1 Numerical Illustration

We now illustrate a numerical exercise with the same input dataset ($p=0.32, q=0.55, r=0.42, h=27.5, A=825.5, Vw=2.5$) to find the EOQ, say Q^* , that minimizes a linear total expected cost function $TEC(Q)$ already used in the QISB

Figure 2: TEC vs Q for QISL with Lost Sales

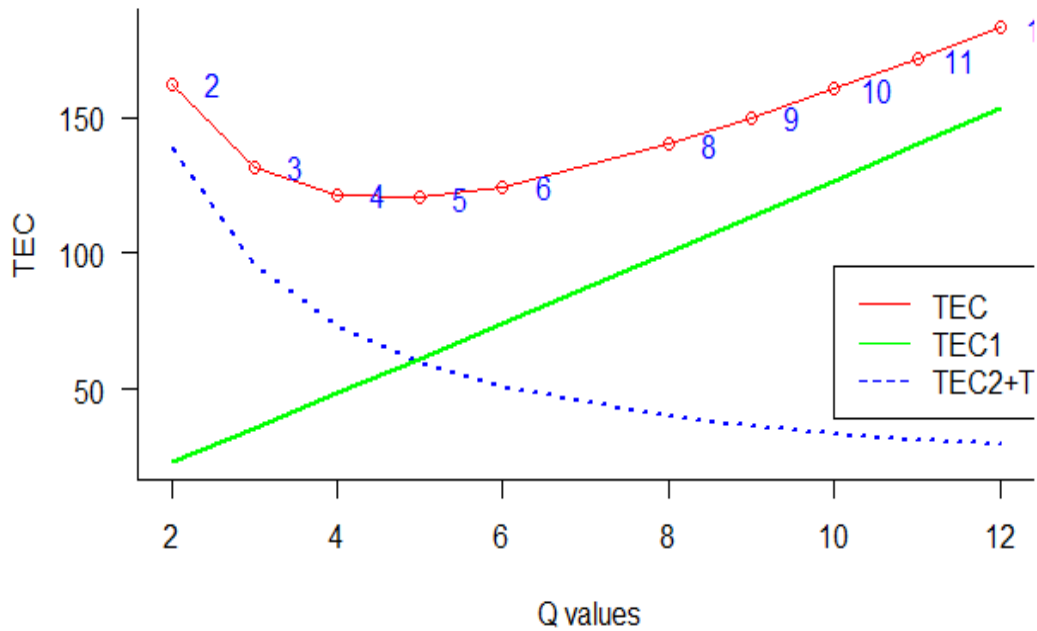


Figure 2: Q values vs TEC, $p=0.32$, $q=0.55$, $r=0.42$ and holding, ordering, and waiting costs ($h=27.5$; $A=825.5$; $V_w=2.5$)

case: $TEC(Q) = \bar{I}h + MRR(A) + \bar{W}_q V_w$

The corresponding values of the TEC are computed, and the resulting Inventory cost vs (MRR cost + Wait cost) and TEC are drawn in Figure 2. The minimum $TEC(Q^*=5) = 120.5$ for $Q=5$.

5. CONCLUSION

Two types of queuing inventory systems, the first case with back-orders, i.e., QISB and the second case with lost customers, i.e., QISL, were studied. In each case, a single server queue of type Geo/Geo/1 is associated with a warehouse that has adopted a threshold policy $TP = [(0, Q), 1]$. The peculiarity is that every Q unit order has been triggered provided that the queue length is at least one and at the same time the inventory is zero.

Steady-state characteristics and performance measures of QISB and QISL were investigated. In each system, a server serves one inventory unit for each customer when completing his service. One can extend these ideas to QIS attached to any repository with (r, Q) policy where $r > 0$ or with (s, S) policy.

Both cases are introduced as a new introduction in the literature using the concept of discrete-time queuing on a time axis divided into an unlimited number of fixed-length intervals called slots. A traditional TEC function is constructed as an implicit function of the order quantity Q and the EOQ is calculated graphically using the same input values for each QISB and QISL case.

The main motivation for favoring QISB and QISL is that each model is a discrete time queue inventory system. This article’s methodology can also be extended to other types of QIS with priority, non-priority, and impatient customers associated with inventory systems with ordering policy (s, S) and its variants.

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