On np-injective rings and Modules

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Abstracts: A given R-module M is called a right np-injective module if for any non-nilpotent element α of R, and any right R-homomorphism $f: \alpha R \to M$ can be extended to $R \to M$, if $M = R_R$ is np-injective, then R is a right np-injective ring. A given ring R is called right weakly np-injective if for each non-nilpotent element α of R, there exists a positive integer n such that any right R-homomorphism $f: \alpha^n R \to M$ can be extended to $R \to M$. A given ring R is a right weakly np-injective ring, if $M = R_R$. In the matrix ring, we poved that R is right np-injective, for some $n \geq 2$, if $M_n(R)$ is right Weakly np-injective. So, We extended many of the known properties and characterizations of right np-injective rings and modules. Finally, the main result in np-injective module found that the R-module M is np-injective module if and only if for any $\alpha \notin N(R)$ the short exact sequence $0 \to \alpha R \xrightarrow{\lambda} R \xrightarrow{\delta} R/\alpha R \to 0$ of R-modules, $0 \to Hom_R(R/\alpha R, M) \xrightarrow{\mathcal{L}} Hom_R(R, M) \xrightarrow{\mathcal{F}} Hom_R(\alpha R, M) \to 0$ is also a short exact sequence, where $\mathcal{F}(f) = f\delta$ and $\mathcal{L}(f) = f\lambda$.

Keywords: Np-Injective, Trivial Extention, Non-Nilpotent Element, Annihilator,R-Modules.

1.INTRODUCTION

In this article, every given rings R are associative rings with identity. All R-modules are unital. The right annihilator and the left annihilator of α are denoted by $r_R(\alpha)$ and $l_R(\alpha)$, respectively. So, we denote $N(R)$, $U(R)$, $Y(R_R)$,and ,$J(R)$ as the nilpotent elements, unit elements, right singular points, and Jacobson radical of R, respectively. So, $\mathbb{Z}_n$ and $\mathbb{Z}$ are the set of integers modulo n and integer numbers. Also, the set of all the abelian groups of R-homomorphism from M to N are denoted by $Hom_R(M,N)$[9]. Additionally, if a given ring of scalars R is commutative. Then, $rm = mr$, for all $r \in R, m \in M$. So, we get that for any $\alpha \in N(R)$, for $(\alpha, m) \in S = R \propto M$, there exist $n \in \mathbb{Z}^+$ with $\alpha^n = 0$, then $(\alpha, m)^{n+1} = (\alpha^{n+1}, (n+1)\alpha^n m) = (0,0)$, for every $\alpha \notin N(R)$ and for every $m \in M$. Thus, the set of all nilpotent elements in $R \propto M$ is given by: $N(R \propto M) = \{(\alpha, m) | \alpha \in N(R) \ and \ m \in M\}$. In adition, if any principal right ideal I of R and any right R-homomorphism $g: I \to M$, there exists y in a R-module M such that $g(\beta) = \beta y$ for all β in I, then M is called p-injective which is defined by Ming in [7]. In[5], Yue Chi Ming generalized p-injective, which is np-injective. He has defined that a right R-module M is called right np-injective if for any $\alpha \notin N(R)$, any R-homomorphism $f: \alpha R \to M$ can be extended to $R \to M$, or equivalently, there exists $m \in M$ such that $f(x) = mx$, for all $x \in \alpha R$. So, the ring R is called right np-injective if R_R is np-injective. Wei and Chen are defined weakly np-injective in [3]. A right R-module M is called weakly np-injective if for any $\alpha \notin N(R)$, there exists a positive integer n and any right R-homomorphism $f: \alpha^n R \to M$ can be extended to $R \to M$. Or equivalently, there exists $m \in M$ such that $f(x) = mx$, for all $x \in \alpha^n R$. It is easy to check that every right np-injective module is right weakly np-injective. If R_R is weakly np-injective, then R is a right weakly np-injective ring [3]. On the other hand, Wei and Chen are generalized p-injective to nil-injective. They have defined that a right R-module M is called nil-injective, if for any R-homomorphism $f\alpha R \to M$ can be extended to $R_R \to M$, or equivalently, there exists $m \in M$ such that $f(x) = mx$, for all $x \in \alpha R$. So, the ring R is called right nil-injective if R_R is nil-injective. Furthermore, right reduced ring and p-injective ring are right nil-injective, but right nil-injective is not right p-injective in general by[8]. Wei and Chen are defined Wnil-injective in [8], a right module M is called Wnil-injective if any $0 \neq \alpha \in N(R)$, there exists a non-negative integer n such that $\alpha^n \neq 0$ and any right R-homomorphism $f\alpha^n R \to M$ can be extended to $R \to M$. Or equivalently, there exists $m \in M$ such that $f(x) = mx$ for all $x \in \alpha^n R$. It is easy to check that every right nil-injective module is right Wnil-injective. If R_R is Wnil-injective, then R is called right Wnil-injective ring [8]. Several authors, including [4], [1] and [2], have investigated the concept of nil-injective, Wnil-injective rings and modules.

2. NP-INJECTIVE RINGS

In this part, we look at some fundamental results concerning np-injective and weakly np-injective rings.

Proposition2.1 For a ring R, the following conditions are equal.

$(1)R$ is a right np-injective ring.

$(2)\ l_R(r_R(\alpha)) = R\alpha$, for every $\alpha \notin N(R)$.

$(3)\ r_R(\alpha) \subseteq r_R(\beta)$, where $\alpha, \beta \notin N(R)$, then $\beta R \subseteq \alpha R$.

$(4)\ l_R(r_R(\alpha) \cap \beta R) = l_R(\beta) + \alpha R$, for all $\alpha, \beta \in R$ with $\beta\alpha \notin N(R)$.

(5) If $f: \alpha R \to R, \alpha \notin N(R)$, is R-linear, then $f(\alpha) \in R\alpha$.

Proof. $(1 \Longrightarrow 2)$ For any $\alpha \notin N(R)$. Let $k\alpha \in R\alpha$, then $k\alpha z = 0$, for each $k \in R$ and $z \in r_R(\alpha)$. This implies $k\alpha \in l_R(z)$, for each $z \in r_R(\alpha)$, yielding $R\alpha \subseteq l_R\big(r_R(\alpha)\big)$. On the other hand, if $x \in l_R\big(r_R(\alpha)\big)$, then $\alpha y = 0$, for each $y \in r_R(\alpha)$. Thus, $xy = 0$. then $r_R(\alpha) \subseteq r_R(x)$. So, let $f \alpha R \to R$ such that $f(\alpha r) = xr$, for each $r \in R$ is a well-defined R-linear map as $\alpha r = \alpha r'$ implies that $\alpha(r - r') = 0$, that is $(r - r') \in r_R(\alpha) \subseteq r_R(x)$. So, $f(\alpha r) = xr = xr' = f(\alpha r')$. Since R is np-injective, there is $\gamma \in R$ such that $f(\alpha r) = \gamma\alpha r$, for all $\alpha r \in \alpha R$. Hence, $xr = \gamma\alpha r$, for all $r \in R$. More specifically, for $r = 1$, we get $x = c\alpha$ and this gives $x \in R\alpha$. Therefore, $l_R\big(r_R(\alpha)\big) \subseteq R\alpha$. Hence, $l_R\big(r_R(\alpha)\big) = R\alpha$, for each $\alpha \notin N(R)$.

$(2 \Longrightarrow 3)$ Suppose $\alpha, \beta \notin N(R))$ such that $r_R(\alpha) \subseteq r_R(\beta)$. Then, $l_R\big(r_R(\beta)\big) \subseteq l_R\big(r_R(\alpha)\big)$. Therefore, $R\beta = l_R\big(r_R(\beta)\big) \subseteq l_R\big(r_R(\alpha)\big) = R\alpha$.

$(3 \Longrightarrow 4)$ First, $(l_R(\beta) + R\alpha) \subseteq l_R(\beta R \cap r_R(\alpha))$ as $x \in (l_R(\beta) + R\alpha)$ implies that $x = y + k\alpha$, where $y\beta = 0$. Now, we must show $x \in l_R(\beta R \cap r_R(\alpha))$. Then, $x(\beta R \cap r_R(\alpha)) = 0$. Therefore, $(\mathrm{y} + \mathrm{k}\alpha)(\beta R \cap r_R(\alpha)) = 0$. We have, $(\mathrm{y} + \mathrm{k}\alpha)(\beta R \cap r_R(\alpha)) = \{(y + k\alpha)\beta t | \alpha\beta t = 0, t \in R\} = \{y\beta t | t \in R\} = \{0\}$. Then, $x \in l_R(\beta R \cap r_R(\alpha))$. Thus, $(l_R(\beta) + R\alpha) \subseteq l_R(\beta R \cap r_R(\alpha))$. Now, let $x \in l_R(\beta R \cap r_R(\alpha))$, then $x(\beta R \cap r_R(\alpha)) = 0$. This means that $x\beta t = 0$ and $\alpha\beta t = 0$, for all $t \in R$. So, whenever $t \in r_R(\alpha\beta), t \in r_R(x\beta)$, showing that $r_R(\alpha\beta) \subseteq r_R(x\beta)$ and so $Rx\beta \subseteq R\alpha\beta$ (by (3)). This implies that, $x\beta = p\alpha\beta$, for some $p \in R$, yielding $(x - p\alpha) \in l_R(\beta)$, that is $x \in l_R(\beta) + R\alpha$. Thus, $l_R(\beta R \cap r_R(\alpha)) \subseteq (l_R(\beta) + R\alpha)$. Hence, $l_R\big(\beta R \cap r_R(\alpha)\big) = l_R(\beta) + R\alpha$.

$(4 \Longrightarrow 5)$ Let $f: \alpha R \to R,\ \alpha \notin N(R)$ be R-homomorphism. Then, $f(\alpha) = \delta$, for some $\delta \in R$. We aim to demonstrate that, $\delta \in R\alpha$. Now, let $x \in r_R(\alpha)$. Then, we get, $0 = f(\alpha x) = f(\alpha)x = \delta x$. So, $r_R(\alpha) \subseteq r_R(\delta)$. This indicates that, $l_R(r_R(\delta)) \subseteq r_R(r_R(\alpha))$. So, $d \in l_R(r_R(\delta)) \subseteq l_R(r_R(\alpha))$. But, $l_R(r_R(\alpha)) = R\alpha$. We set,

$\beta = 1$ in (4). Therefore, $l_R(R \cap r_R(\alpha)) = l_R(r_R(\alpha)) = l_R(r_R(\alpha) \cap R) = l_R(1) + R\alpha = R\alpha$. Hence, $d \in R\alpha$.

$(5 \Longrightarrow 1)$Let $f: \alpha R \to R$ be R-homomorphism with $f(\alpha) \in R\alpha$. Then, $f(\alpha) = c\alpha$, for some $c \in R$. This proves (1).

Theorem 2.2 Assume R is a right np-injective ring and $\mathrm{R}\alpha_1 \oplus \mathrm{R}\alpha_2 \oplus \ldots \oplus \mathrm{R}\alpha_n$ is a direct sum, for each $\alpha_i \notin \mathrm{N}(\mathrm{R})$, for $\mathrm{i} = 1,2,\ldots,\mathrm{n}$. Then, any R-homomorphism $\mathrm{f}\alpha_1\mathrm{R} \oplus \alpha_2\mathrm{R} \oplus \ldots \oplus \alpha_n\mathrm{R} \to \mathrm{R}$ extends to $\mathrm{g}: \mathrm{R} \to \mathrm{R}$.

Proof. Suppose that $f\alpha_1 R + \alpha_2 R + \ldots + \alpha_n \mathrm{R} \to R$ is an R-homomorphism. Since $f|\alpha_i R$ is a right np-injective, then $f|\alpha_i R: \alpha_i R \to R$ is given by $f(\alpha_i) = \alpha_i\beta_i$, for some $\beta_i \in R$ and for $i = 1,2,\ldots,n$. Obviously, $\alpha_1 + \cdots + \alpha_n \notin N(R)$, where $\alpha_i \notin N(R)$, for $i = 1,2,\ldots,n$. Since R is np-injective, then the R-homomorphism $h(\alpha_1 + \alpha_2 + \ldots + \alpha_n)\mathrm{R} \to R$ defined by $h(\alpha_1 + \cdots + \alpha_n) = (\alpha_1 + \cdots + \alpha_n)\beta$. Therefore, h can be extented to R-homomorphism g such that $g: R \to R$. Since $(\beta_1 + \ldots + \beta_n)R \subseteq \beta_1 R + \ldots + \beta_n R$, then:

$$f(\alpha_1 + \cdots + \alpha_n) = h(\alpha_1 + \cdots + \alpha_n)$$

$$f(\alpha_1) + f(\alpha_2) + \cdots + f(\alpha_n) = h(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$$

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n = (\alpha_1 + \alpha_2 + \cdots + \alpha_n)\beta$$

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n = \alpha_1\beta + \alpha_2\beta + \cdots + \alpha_n\beta$$

Since $R\alpha_1 \oplus R\alpha_2 \oplus \ldots \oplus R\alpha_n$ is direct sum, which implies that $\alpha_1\beta_1 = \alpha_1\beta$, $\alpha_2\beta_2 = \alpha_2\beta$, ..., $\alpha_n\beta_n = \alpha_n\beta$ that is $\alpha_i\beta_i = \alpha_i\beta$, for each $\alpha_i \notin N(R)$, for $i = 1,2,\ldots,n$. Hence, $f: R\alpha_1 + R\alpha_2 + \cdots + R\alpha_n \to R$ can be extended as $g: R \to R$.

Theorem2.3 Let R be a right np-injective ring, the sum $\beta_1 R \oplus \beta_2 R \oplus \ldots \oplus \beta_n R$ be direct, $\beta_i \notin N(R)$, for each $i = 1,2,\ldots,n$, $D = \beta_1 R + \ldots + \beta_k R$ and $T = \beta_{k+1} R + \ldots + \beta_n R$, $1 \leq k \leq n$. Then, $l_R(D \cap T) = l_R(D) + l_R(T)$.

Proof. Let $x \in (l_R(D) + l_R(T))$, then $x = y + z$, for Dome $y \in l_R(D), z \in l_R(T)$. Now, for any $k \in D \cap T$, then $xk = (y + z)k = yk + zk = 0 + 0 = 0$. Therefore, $(l_R(D) + l_R(T)) \subseteq l_R(D \cap T)$. Now, let $x \in l_R(D \cap T)$, then $\chi: D + T \to R$ given by $\chi(s + t) = sx$ is an R-homomorphism. By using Theorem 2.2, can be extended. Therefore, $\chi(s + t) = \alpha(s + t)$, for some $\alpha \in R$ and, for each $(s + t) \in (D + T)$. Now, $\alpha t = \chi(0 + t) = 0.x = 0$, so $\alpha \in l_R(T)$. Therefore, $\chi(s + t) = a(s + t) = sx$, yields $(x - a)s = 0$, that is, $(x - a) \in l_R(D)$ and so $(x - a) + a = x$. Then, $x \in (l_R(D) + l_R(T))$. Thus, $l_R(D \cap T) \subseteq l_R(D) + l_R(T)$. As a result, $l_R(D \cap T) = l_R(D) + l_R(T)$.

Theorem2.4 Assume R is a right np-injective ring. If the sums $\alpha R \oplus \beta R$ and $R\alpha \oplus R\beta$ are both direct sum, then $l_R(\alpha) + l_R(\beta) = R$, for $\alpha, \beta \notin N(R)$.

Proof. We have, $(l_R(\alpha) + l_R(\beta)) \subseteq R$. We define, $f: (\alpha + \beta)R \to R$ by $f((\alpha + \beta)k) = \beta k$. We must demonstrate that f is well-defined. If $k, k' \in R$, then $(\alpha + \beta)k = (\alpha + \beta)k'$, this implies that $\alpha(k - k') = \beta(k' - k) \in \alpha R \cap \beta R$. But, $\alpha R \oplus \beta R$ is a direct, so $\alpha(k - k') = \beta(k' - k) = 0$, yielding

$\beta k' = \beta k$. Thus, f is well-defined. We know that R is a right np-injective, then f can be extended on R. Therefore, $f(\alpha+\beta) = \gamma(\alpha+\beta)$, for some $\gamma \in R$. Then, $\beta = \gamma(\alpha+\beta)$, and so $\gamma\alpha = (1-\gamma)\beta \in \alpha R \cap \beta R = \{0\}$ implies that $\gamma \in l_R(\alpha)$, $1-\gamma \in l_R(\beta)$. This indicates that $1 \in l_R(\gamma) + l_R(\beta)$. Whence, $l_R(\gamma) + l_R(\beta) = R$.

Proposition2.5 An integral domain is right np-injective if and only if it is a division ring.

Proof. Assume that a given ring R is a division ring, then for any $\alpha \in R$, either $\alpha = 0$ or α is a unit. If α is unit, then $r_R(\alpha) = 0$. Since $\alpha^{-1} \in R$. Thus, $R = R\alpha$. Therefore, by Theorem 2.1, $l_R(r_R(\alpha)) = R\alpha$ and so R is a right np-injective.

Conversely, let $0 \neq \alpha \notin N(R)$ and let $f: \alpha R \to R$ be given by $f(\alpha k) = k$, f is a well-defined since R is an integral domain. Also f is R-linear. Now, by hypothesis, f can be extended to an R-linear map $g: R \to R$. Hence, $\alpha g(1) = g(\alpha) = f(\alpha) = 1$. This shows that α is a unit of R. Hence, R is a division ring.

Proposition2.6 Assume that a given ring R is a local right np-injective ring. Then, $\alpha R \cap \beta R \neq 0$, for any $\alpha, \beta \notin N(R)$.

Proof. Suppose that $\alpha R \cap \beta R = 0$ and the R-homomorphism $f: (\alpha+\beta)R \to R$ defined by $f((\alpha+\beta)k) = \beta k$. Let $(\alpha+\beta)k = (\alpha+\beta)k'$. So, $\alpha(k-k') = \beta(k'-k) = 0$, yielding $\beta k' = \beta k$. Thus, f is a well-defined. Since R is right np-injective, then f can be extended to R-homomorphism R to R. Therefore, $f((\alpha+\beta)) = (\alpha+\beta)\gamma$, for some $\gamma \in R$. Thus, $\beta = (\alpha+\beta)\gamma$. Since R is local, then either γ or $1-\gamma$ is a unit, but $0 = \alpha\gamma = \beta(1-\gamma) \in (\alpha R \cap \beta R) = \{0\}$. Thus, $\alpha = 0$ or $\beta = 0$, it is a contradiction. Hence, $\alpha R \cap \beta R \neq 0$, for any $\alpha, \beta \notin N(R)$.

Proposition2.7 Every non-zero divisor of R is invertible in R. If R_R is a right np-injective ring.

Proof. Assume γ is a non-zero divisor of R. Define $F: \gamma R \to R$ by $F(\gamma\alpha) = \alpha$, for all $\alpha \in R$. Then, F is a well-defined right R-homomorphism. We know that R_R is a right np-injective ring, then there exists an R-homomorphism $H: R \to R$, for every a right R-homomorphism $I: R \to R$ such that $H \circ I = F$. If $H(1) = u \in R$, then $1 = F(c) = H \circ I(c) = H(c) = cH(1) = cu$. Then, $\gamma = \gamma u \gamma$, which yields $\gamma(1-u\gamma) = 0$, whence $u\gamma = 1$. Therefore, γ is invertible in R.

Proposition2.8 Assume R is a principal ideal ring such that R has no non-zero nilpotent elements. If R is a right np-injective ring, then $J(R) = Y(R_R)$.

Proof. Let $x \in Y(R_R)$. Since $Y(R_R)$ is a two-sided ideal, then, for each $\gamma \in R$. $\gamma x \in Y(R_R)$. Then, $r_R(\gamma x)$ is essential ideal of R, let $y \in (r_R(\gamma x) \cap r_R(1-\gamma x))$ implies $y\gamma x = 0$ and $y(1-\gamma x) = 0$ gives $y = 0$, this is a contradiction. Thus, $r_R(1-\gamma x) = 0$ implies that $l_R(r_R(1-\gamma x)) = l_R(0)$, that is, $R(1-\gamma x) = R$. Thus, $y(1-\gamma x) = 1$, for some $y \in R$, so $1-\gamma x$ has left inverse, for all $\gamma \in R$. By [Proposition 6.1.8.,[6]] , $x \in J(R)$. Thus, $Y(R_R) \subseteq J(R)$. Now, let $x \in J(R)$. We want to show $x \in Y(R_R)$, that is $r_R(x) \cap \alpha R = 0$ implies $\alpha R = 0$. Now, let $\alpha \notin N(R)$. Then, $r_R(x) \cap \alpha R = 0$ indicates that $l_R(r_R(x) \cap \alpha R) = r_R(0)$. By Proposition 2.1, $xR + l_R(\alpha) = R$, so we can write $xk + p = 1$, which implies $p = 1 - xk \in l_R(\alpha)$.Since $x \in J(R)$. By Proposition [Proposition 6.1.8.,[6]], $1-xk$ is left invertible that is $pt = 1 \in l_R(\alpha)$, for $t \in R$, yielding $r_R(\alpha) = R$ and so $\alpha = 0$. Thus, $\alpha R = 0$. Hence, $J(R) \subseteq Y(R_R)$. Therefore, $J(R) = Y(R_R)$.

Proposition2.9 Let R be a right np-injective and let $\alpha, \beta \notin N(R)$.

(1) If $R\beta$ embeds in $R\alpha$, then βR is an image of αR.

(2) If $R\alpha$ is an image of $R\beta$, then αR embeds in βR.

(3) If $R\beta \cong R\alpha$, then $\alpha R \cong \beta R$.

Proof. (1) Let $F: \beta R \rightarrow \alpha R \subseteq R$ be an R-monomorphism, then $F(\beta) = \alpha u$, for some $u \in R$. Since R is a right np-injective, then $F(\beta) = v\beta$, for some $v \in R$. So, $v\beta = u\alpha$, for some$u \in R$. Let $H: R\alpha \rightarrow R\beta$ defined by $H(k\alpha) = (v\beta)k$, for every $k \in R$. Then H is a well-defined as $k\alpha = k'\alpha$ implies $uk\alpha = uk'\alpha$, yielding $v\beta k = v\beta k'$, that is $H(k\alpha) = H(k'\alpha)$. Now, we show that H is onto. Let $x \in r_R(v\beta)$, then we have $v\beta x = 0$, which implies that $\alpha x = 0$. Since F is a monomorphism, then $F(\beta x) = 0$, which implies $\beta x = 0$. This shows that $x \in r_R(\beta)$. Therefore, $r_R(v\beta) \subseteq r_R(\beta)$, which implies $l_R(r_R(\beta)) \subseteq l_R(r_R(v\beta))$. So, $R\beta \subseteq v\beta R$, that is $\beta = v\beta k = H(k\alpha)$, thus H is onto. Hence, $R\beta$ is an image of $R\alpha$.

(2) Suppose $F\beta R \rightarrow \alpha R \subseteq R$ is an onto R-homomorphism, let v, u and H be as in , that is $HR\alpha \rightarrow R\beta$ defined by $H(k\alpha) = k(v\beta), v\beta = u\alpha$, for some $u \in R$. Now, $\alpha = F(\beta s) = \sigma(\beta)s = v\beta s$, for some $s \in R$. If $H(k\alpha) = 0$, then $v\beta k = 0$ and hence $k\alpha = kv\beta s = kv\beta . s = 0. s = 0$, thus H is one-one. Hence, $R\beta$ embeds in $R\alpha$.

(3) Combining (1) and (2), we get (3).

3. MAIN RESULT

Our interest here is in right np−injective rings. So, we find some properties and characterizations about np-injective rings. On the other hand, we discuss an annihilator condition of trivial extensions. Assume R is a ring, the trivial extension $S = R \propto R = \{(\alpha, \beta) | \alpha, \beta \in R\}$ is a ring with addition defined componentwise and multiplication defined by: $(\alpha, \beta)(\gamma, d) = (\alpha\gamma, \alpha d + \beta\gamma)$. Furthermore, it can be notice that for any $\alpha \in N(R)$, then $(\alpha, \beta) \in S = R \propto R$, there exists $n \in Z^+$ such that $\alpha^n = 0$, then $(\alpha, \beta)^{(n+1)} = (\alpha^{(n+1)}, (n+1)\alpha^n\beta) = 0$ where R is a commutative ring.

Theorem3.1 The following ring R statements are equal:

(1) R is right np-injective ring.

(2) (a) If $\{\alpha_i R\}_{i=1}^n$ is a family of α_i R of R, for each $\alpha_i \notin N(R)$. Then $l_R(\cap_{i=1}^n \alpha_i R) = \sum_{i=1}^n l_R(\alpha_i R)$.

(b) $l_R(r_R(R\alpha)) = R\alpha$, for every $\alpha \notin N(R)$.

Proof. $(1 \Longrightarrow 2a)$ Let $x \in \sum_{i=1}^n l_R(\alpha_i R)$, for each $\alpha_i \notin N(R)$, where $i = 1,2,\dots,n$. Then, $x = y_1 + y_2 + \cdots + y_n$, for some $y_i \in l_R(\alpha_i R)$. Now, for any $k \in l_R(\bigcap_{i=1}^n \alpha_i R)$, $xk = (y_1 + y_2 + \cdots + y_n)k = y_1 k + y_2 k + \cdots + y_n k = 0 + 0 + \cdots + 0 = 0$. Thus, $x \in l_R(\bigcap_{i=1}^n \alpha_i R)$. Therefore, $\sum_{i=1}^n l_R(\alpha_i R) \subseteq l_R(\bigcap_{i=1}^n \alpha_i R)$. Now, we will prove $l_R(\alpha_1 R \cap \alpha_2 R) \subseteq l_R(\alpha_1 R) + l_R(\alpha_2 R)$. Let $\gamma \in l_R(\alpha_1 R \cap \alpha_2 R)$ and $f\alpha_1 R + \alpha_2 R \rightarrow R$ given by $f(\alpha) = \alpha$, for each $\alpha \in \alpha_1 R$ and $f(\beta) = (1+\gamma)\beta$, for each $\beta \in \alpha_2 R$. Obviously, the two expressions for f agree on $\alpha_1 R \cap \alpha_2 R$, this R-homomorphism is well defined.

Since R is np-injective, then by Proposition 2.1, there is $x \in R$ such that $f(r) = xr$, for each $r \in \alpha_i R$. If $r = \alpha$, then $\alpha = f(\alpha) = x\alpha$. Then, $(x-1)\alpha = 0$. Therefore, $x - 1 \in l_R(\alpha_1 R)$. If $r = \beta$, then $f(\alpha) = (1+\gamma)\beta = x\beta$. Therefore, $(1+\gamma-x)\beta = 0$. As a result, $(1+\gamma-x) \in l_R(\alpha_2 R)$. Hence, $\gamma = (x-1) + (1+\gamma-x) \in l_R(\alpha_1 R) + l_R(\alpha_2 R)$. We obtain, $l_R(\alpha_1 R) + l_R(\alpha_2 R) = l_R(\alpha_1 R \cap \alpha_2 R)$. By induction, assume that $l_R(\cap_{i=1}^{n-1} \alpha_i R) \subseteq \sum_{i=1}^{n-1} l_R(\alpha_i R)$. We have to prove that $l_R(\cap_{i=1}^{n} \alpha_i R) \subseteq \sum_{i=1}^{n} l_R(\alpha_i R)$. Let $\gamma \in l_R(\cap_{i=1}^{n} \alpha_i R)$ and $f: \alpha_i R \to R$ given by $f(\alpha_i) = \alpha_i$, where $i = 1,2,\dots,n$ and $f(\beta) = (1+\gamma)\beta$, for all $\beta \in \alpha_n R$. We know that R is np-injective then by Proposition 2.1, there is $x \in R$ such that $f(r) = xr$, for each $r \in \alpha_i R$, where $i = 1,2,\dots,n$. If $r = \alpha_i$, then $\alpha_i = f(\alpha_i) = x\alpha_i$. Then, $(x-1)\alpha_i = 0$. Therefore, $x - 1 \in l_R(\cap_{i=1}^{n-1} \alpha_i R) \subseteq \sum_{i=1}^{n-1} l_R(\alpha_i R)$. If $r = \beta \in \alpha_i R$, then $f(\beta) = (1+\gamma)\beta = x\beta$. Thus, $(1+\gamma-x)\beta = 0$. Therefore, $(1+\gamma-x) \in l_R(\alpha_n R)$. Hence, $\gamma = (x-1) + (1+\gamma-x) \in \sum_{i=1}^{n-1} l_R(\alpha_i R) + l_R(\alpha_n R)$. We obtain, $l_R(\cap_{i=1}^{n} \alpha_i R) = \sum_{i=1}^{n} l_R(\alpha_i R)$.

$(1 \Longrightarrow 2b)$ For any $\alpha_i \notin N(R)$. Let $z \in r_R(R\alpha_i)$, then $k\alpha_i z = 0$, for each $k \in R$. This implies $k\alpha_i \in l_R(z)$, for each $z \in r_R(R\alpha_i)$, yielding $R\alpha_i \subseteq l_R(r_R(R\alpha_i))$. On the other hand, if $\beta \in l_R(r_R(R\alpha_i))$, then $f\alpha_i R \to \beta R \subseteq R$ such that $f(\alpha_i r) = \beta r$, for each $r \in R$ is a well-defined R-homomorphism. Since R is right np-injective, there is an $x \in R$ such that $f(\alpha_i r) = x\alpha_i r$, for all $\alpha_i r \in \alpha_i R$. Hence, $\beta r = x\alpha_i r$, for all $r \in R$. More specifically, for $r = 1$, we have $\beta = x\alpha_i$ and this gives $\beta \in R\alpha_i$. Therefore, $l_R(r_R(R\alpha_i)) \subseteq R\alpha_i$. Hence, $l_R(r_R(R\alpha_i)) = R\alpha_i$, for each $\alpha_i \notin N(R)$.

$(2 \Longrightarrow 1)$ For any $\alpha_i \notin N(R)$, where $i = 1,\dots,n$. Suppose that $f\alpha_i R \to R$ is R-homomorphism. Then, $f(\alpha_i) = f(1)\alpha_i$. Then, $f(1)\alpha_i \cdot r_R(R\alpha_i) = 0$. By using(2b), gives $f(1)\alpha_i \in l_R(r_R(R\alpha_i)) = R\alpha_i$. Let $f(1) = x$, for some $x \in R$. Therefore, the R-homomorphism $f\alpha_i R \to R$ given by $f(\alpha_i r) = f(\alpha_i)r = f(1)\alpha_i r = x\alpha_i r$, for all $\alpha r \in \alpha R$. Thus, f can be extended from R to R. Hence, R is right np-injective ring.

Theorem 3.2 Let R be a ring and $S = R \propto R$, the trivial extension of R. Then, R is right np-injective ring if and only if one of the conditions listed below is satisfies:

1. $l_S(r_S(0,\alpha)) = S(0,\alpha)$.
2. $l_S(r_S(\alpha,0)) = S(\alpha,0)$.
3. $l_S(r_S(\alpha,\alpha)) = S(\alpha,\alpha)$.

Proof. (1). Let $(\beta,c) \in (0,\alpha)S$. For some $(x,y) \in S$, $(\beta,c) = (x,y)(0,\alpha) = (0,x\alpha)$. Then, $\gamma = x\alpha \in R\alpha = l_R(r_R(\alpha))$. Therefore, $0 = x\alpha.x_1$, for all $x \in R$ and $x_1 \in r_R(\alpha)$. Now, suppose that $(\beta,\gamma) \notin l_S(r_S(0,\alpha))$. Then, there exists $(x_1,y_1) \in r_S(0,\alpha)$ such that $(0,x\alpha)(x_1,y_1) = (0,x\alpha x_1) \neq (0,0)$. Then, $x\alpha x_1 \neq 0$. By Theorem 2.1, $x\alpha \notin l_R(r_R(\alpha)) = R\alpha$, which is contradiction. Therefore, $S(0,\alpha) \subseteq l_S(r_S(0,\alpha)$. On the other hand, let $(\beta,\gamma) \in l_S(r_S(0,\alpha))$, then $r_S(0,\alpha) \subseteq r_S(\beta,\gamma)$. Now, $(0,\alpha)(0,1) = (0,0)$ implies that $(0,1) \in r_S(0,\alpha) \subseteq r_S(\beta,\gamma)$. So, $(0,\beta) = (\beta,\gamma)(0,1) = (0,0)$ and thus $\beta = 0$. If $x \in r_R(\alpha)$, then $\alpha x = 0$, which implies $(0,\alpha)(x,0) = (0,0)$, that is $(x,0) \in r_S(0,\alpha) \subseteq r_S(\beta,\gamma)$. Which gives $(\beta x, \gamma x) = (\beta,\gamma)(x,0) = (0,0)$ and hence $\gamma x = 0$, that is $x \in r_S(\gamma)$. Therefore, $r_S(\alpha) \subseteq r_S(\gamma)$. Thus, $\gamma \in l_R(r_R(\alpha)) = R\alpha$. So, let $\gamma = t\alpha$, for some $t \in R$. Then, $(\beta,\gamma) = (0,t\alpha) = (t,0)(0,\alpha) \in S(0,\alpha)$. Hence, $l_S(r_S(0,\alpha)) = S(0,\alpha)$.

Conversely, let $\beta \in l_R(r_R(\alpha))$, then $r_R(\alpha) \subseteq r_R(\beta)$. If $(x,y) \in r_R(0,\alpha)$, then $(0,\alpha x) = (0,\alpha)(x,y) = (0,0)$ and so $\alpha x = 0$. That is, $x \in r_R(\alpha) \subseteq r_R(\beta)$, yielding $(0,\beta)\ (x,y) = (0,0)$. Since $(x,y) \in r_R(0,\alpha)$, then $r_R(0,\alpha) \subseteq r_R(0,\beta)$. So we have $(0,\beta) \in l_R(r_R(0,\alpha)) = S(0,\alpha)$, that implies $(0,\beta) = (x,y)(0,\alpha) = (0,x\alpha)$, for some $(x,y) \in S$. Thus, $\beta = x\alpha \in R$. Hence, $l_R(r_R(\alpha)) = R\alpha$.

(2). Let $(\beta,\gamma) \in l_S(r_S(\alpha,0))$. Firstly, we need to show that, $(\beta,\gamma) = (p,q)(\alpha,0)$, for some $(p,q) \in S$. We have, $(\beta,\gamma) \in l_S(r_S(\alpha,0))$ implies that $r_S(\alpha,0) \subseteq r_S(\beta,\gamma)$. We will show that $r_R(\alpha) \subseteq r_R(\beta)$ and $r_R(\alpha) \subseteq r_R(\gamma)$. We take $x \in r_R(\alpha)$, then $\alpha x = 0$, which implies $(\alpha,0)(x,0) = (0,0)$ and so $(x,0) \in r_S(\alpha,0) \subseteq r_S(\beta,\gamma)$. Finally, the equation $(\beta,\gamma)(x,0) = (0,0)$, gives $\beta x = 0$ and $\gamma x = 0$, yielding $r_R(\alpha) \subseteq r_R(\beta)$ and $r_R(\alpha) \subseteq r_R(\gamma)$. Thus, $l_R(r_R(\beta)) \subseteq l_R(r_R(\alpha))$ and $l_R(r_R(\gamma)) \subseteq l_R(r_R(\alpha))$. This further implies that $R\beta \subseteq R\alpha$ and $R\gamma \subseteq R\alpha$. Therefore, $(\beta,\gamma) = (t\alpha,k\alpha)$, for some $t,k \in R$ and $(\beta,\gamma) = (t,k)(\alpha,0) \in S(\alpha,0)$. Hence, $l_S(r_S(\alpha,0)) = (\alpha,0)S$.

Conversely, let $x \in l_R(r_R(\alpha))$. Then, $r_R(\alpha) \subseteq r_R(x)$. If $(p,q) \in r_S(\alpha,0)$, then $p \in r_R(\alpha) \subseteq r_R(x)$ and $q \in r_R(\alpha) \subseteq r_R(x)$ and so we have, $(x,0)(p,q) = (0,0)$, that is, $(p,q) \in r_S(x,0)$. Thus, $r_S(\alpha,0) \subseteq r_S(x,0)$, which implies $l_S(r_S(x,0)) \subseteq l_S(r_S(\alpha,0))$, that is, $S(x,0) \subseteq S(\alpha,0)$, so we can write $(1,1)(x,0) = (x,0) = (u,v)(\alpha,0) = (u\alpha,v\alpha)$, for some $(u,v) \in S$, yielding $x = u\alpha$, thus $x \in R\alpha$. Hence, $l_R(r_R(\alpha)) \subseteq R\alpha$. Therefore, $l_R(r_R(\alpha)) = R\alpha$.

(3). Let $(\beta,\gamma) \in l_S(r_S(\alpha,\alpha))$, then $r_S(\alpha,\alpha) \subseteq r_S(\beta,\gamma)$. If $x \in r_R(\alpha)$, then $(\alpha,\alpha)(x,0) = (0,0)$, that is, $(x,0) \in r_S(\alpha,\alpha) \subseteq r_S(\beta,\gamma)$, then $(\beta,\gamma)(x,0) = (0,0)$ implies $\beta x = 0, \gamma x = 0$, that is, $x \in l_R(\beta)$ and $x \in l_R(\gamma)$, which implies $l_R(r_R(\beta)) \subseteq l_R(r_R(\alpha))$ and $l_R(r_R(\gamma)) \subseteq l_R(r_R(\alpha))$, so we can write $\beta = t\alpha$ and $\gamma = k\alpha$. Thus, $(\beta,\gamma) = (t,k)(\alpha,\alpha)$. Therefore, $l_S(r_S(\alpha,\alpha)) = S(\alpha,\alpha)$.

Conversely, let $x \in l_R(r_R(\alpha))$, then $r_R(\alpha) \subseteq r_R(x)$. Let $(p,q) \in r_S(\alpha,\alpha)$, which implies $\alpha p = 0, \alpha q = 0$, so we have $p \in r_R(\alpha) \subseteq r_R(x)\ and\ q \in r_R(\alpha) \subseteq r_R(x)$, that is, $xp = 0\ and\ xq = 0$, which implies $(x,x)(p,q) = (0,0)$, that is, $(p,q) \in r_S(x,x)$. Thus, $r_S(\alpha,\alpha) \subseteq r_S(x,x)$, which implies $l_S(r_S(x,x)) \subseteq l_S(r_S(\alpha,\alpha))$, then we have $(x,x) = (k,t)(\alpha,\alpha)$, for some $(k,t) \in S$, yielding $x = k\alpha,\ x = k\alpha + t\alpha$. Therefore, $x \in R\alpha$. Thus, $l_R(r_R(\alpha)) = R\alpha$.

Corollary 3.3 Let R be a ring. If $S = R \propto R$ is right Weakly np-injective, then R is right np-injective.

Proof. Assume $S = R \propto R$. For any $0 \neq \alpha \in R, 0 = (0,\alpha) \in S$. Since $S = R \propto R$ is right Weakly np-injective, then by [Theorem 2.1,[3]] there exists $n > 0$ such that $(0,\alpha)^n \neq 0$ and $l_S(r_S(0,\alpha)^n) = S(0,\alpha)^n$. Since $(0,\alpha)^2 = 0$, it has to be that $n = 1$. So $l_S(r_S(0,\alpha)) = S(0,\alpha)$. It follows from Theorem 3.2, R is right np-injective.

Theorem 3.4 Assume R is a commutative ring, if every right R-homomorphism from $\alpha R + \beta r_R(\alpha)$ to R extends to one from R to R, for any $\alpha \notin N(R)$, Then, $S = R \propto R$ is right np-injective.

Proof. Let R be a commutative ring and $S = R \propto R$. Let $\alpha \notin N(R)$, then $(\alpha,\beta)^n = (\alpha^n, n\alpha^{(n-1)}\beta) = (0,0)$ and $N(S) = \{(\alpha,\beta) \in S | \alpha \in N(R)\ and\ \beta \in R\}$. Suppose, $\overleftarrow{u} = (\alpha,\beta) \in N(S)$ and $f \overleftarrow{u} S \to S$ is a right S-homomorphism. We want to show that f has an extension on S. Let $f(\alpha,\beta) = (p,q) \in S$. Let $g \alpha R + \beta r_R(\alpha) \to R$ be given by $g(\alpha k + \beta\gamma) = pk + q\gamma$ for some $p,k,q,\gamma \in R$. We first show that g is well-

defined. If $\alpha k_1 + \beta\gamma_1 = \alpha k_2 + \beta\gamma_2$, for some $\gamma_1, \gamma_2 \in r_R(\alpha)$, then $\alpha k_1 - \alpha k_2 + \beta\gamma_1 - \beta\gamma_2 = 0$, therefore $\alpha(k_1 - k_2) + \beta(\gamma_1 - \gamma_2) = 0$, then $(\alpha, \beta)(\gamma_1 - \gamma_2, k_1 - k_2) = (0,0)$. Then, $f((\alpha, \beta)(\gamma_1 - \gamma_2, k_1 - k_2)) = f(0,0)$. Therefore, $f(\alpha, \beta)(\gamma_1 - \gamma_2, k_1 - k_2) = (0,0)$. Thus, $(p, q)(\gamma_1 - \gamma_2, k_1 - k_2) = (0,0)$. Then, $p(k_1 - k_2) + q(\gamma_1 - \gamma_2) = (0,0)$. yielding $pk_1 + q\gamma_1 = pk_2 + q\gamma_2$ thus g is well-defined. So, g is a right R-homomorphism as $g((\alpha k + \beta\gamma) + (\alpha k' + \beta\gamma')) = p(k + k') + q(\gamma + \gamma') = (pk + q\gamma) + (pk' + q\gamma') = g(\alpha k + \beta\gamma) + g(\alpha k' + \beta\gamma')$
and $g(t(\alpha k + \beta\gamma)) = g(t\alpha k + t\beta\gamma) = tpk + tq\gamma = t(pk + q\gamma) = tg(\alpha k + \beta\gamma)$. So, there exists $z \in R$ such that $g(w) = zw$, for all $w \in \alpha R + \beta r_R(\alpha)$. Now, $g(\alpha) = z\alpha$ but $g(\alpha) = p$. So, $\alpha z = p$. Let $\varphi \alpha R \to R$ be given by $\varphi(\alpha t) = (\beta z - q)t$. φ is well-defined for if $\alpha t = \alpha t'$, then by definition of g, $\beta z(t - t') = \beta g(1)(t - t') = g(\beta(t - t')) = q(t - t')$, which implies $(\beta z - q)t = (\beta z - q)t'$. Thus, there exists $m \in R$ such that $\alpha u m = \varphi(\alpha u)$, which implies $\alpha u m = \varphi(\alpha u) = (-\beta z + q)u$, for all $u \in R$, yielding $q = \alpha m + \beta z$. Let $\overleftarrow{w} = (z, m) \in S$. Then, $\overleftrightarrow{u}\overleftarrow{w} = (\alpha, \beta)(z, m) = (\alpha z, \alpha m + \beta z) = (p, q) = f(\overleftarrow{w})$. Thus, $f(\overleftarrow{w}) = \overleftrightarrow{u}\overleftarrow{w}$. So, $hS \to S$ given by $h(1) = \overleftarrow{w}$ is the extension of f. Hence S is a right np-injective.

Theorem 3.5 Assume R is a ring.and $S = M_n(R)$ be the matrix ring, for $\alpha \notin N(R)$. Then the followings are true:

(1) $l_S(r_S(E_{n1}\alpha)) = SE_{n1}$ a if and only if $l_R(r_R(\alpha)) = R\alpha$.

(2) If $\mathrm{M_n(R)}$ is right Weakly np-injective, for some $n \geq 2$, then R is right np-injective.

Proof. (1) Let $\beta \in l_R(r(\alpha))$, then $r_R(\alpha) \subseteq r_R(\beta)$. Now, take $(\gamma_{ij}) \in r_S(\alpha E_{n1})$, then

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{12} & . & . & . & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & . & . & . & \gamma_{11} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ . & . & . & . & . & . \\ \gamma_{n1} & \gamma_{n2} & . & . & . & \gamma_{nn} \end{pmatrix} = 0.$$

So, we have $\alpha\gamma_{1i} = 0$, for all $i = 1, 2, \dots, n$. That is, $\gamma_{1i} \in r_R(\alpha) \subseteq r_R(\beta)$, so $\beta\gamma_{1i} = 0$, for all i, yielding $(\beta E_{n1})(\gamma_{1i}) = 0$. Thus, $(\gamma_{ij}) \in r_R(\beta E_{n1})$, hence $r_S(\alpha E_{n1}) \subseteq r_S(\beta E_{n1})$. Therefore, $E_{n1} \in l_S(r_S(\alpha E_{n1})) = S\alpha E_{n1}$. So, we can write $\beta E_{n1} = (d_{ij})\alpha E_{n1}$, where $(d_{ij}) \in S$ which implies $\beta = d_{nn}\alpha \in R\alpha$. Hence, $l_R(r_R(\alpha)) = R\alpha$.

Conversely, Let $\beta = (\beta_{ij}) \in l_S(r_S(\alpha E_{n1}))$ then $r_R(\alpha E_{n1}) \subseteq r_R(B)$. Now, if $i \neq 1$, then $(\alpha E_{n1})E_{ij} = 0$, which implies $E_{ij} \in r_S(\alpha E_{n1}) \subseteq r_S(B)$, thus $E_{ij}B = 0$, that is, $(\beta_{ij})(E_{ij}) = 0$ hence $\beta_{ki} = 0$, for $k = 1, 2, .., n$. So,

$$B = \begin{pmatrix} \beta_{11} & 0 & . & . & . & 0 \\ \beta_{21} & 0 & . & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ . & . & . & . & . & . \\ \beta_{n1} & 0 & . & . & . & 0 \end{pmatrix}.$$

Now, If $\beta \in r_R(\alpha)$, then $\alpha E_{11} \in r_S(\alpha E_{n1}) \subseteq r_S(\beta)$. So, $\beta \in r_R(\beta_{i1})$, for $i = 1,2,..,n$. Thus, $r_R(\alpha) \subseteq r_R(\beta_{i1})$ implies $l_R\left(r_R(\beta_{1j})\right) \subseteq l_R(r_R(\alpha))$, then $\beta_{i1} \in l_R(r_R(\beta_{i1})) \subseteq l_R(r_R(\alpha)) = R\alpha$. So, $\beta_{i1} = t_{i1}\alpha$ with $t_{i1} \in R$, for $i = 1,\ldots,n$. Thus,

$$B = \begin{pmatrix} t_{11}\alpha & 0 & . & . & . & 0 \\ t_{21}\alpha & 0 & . & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ . & . & . & . & . & . \\ t_{n1}\alpha & 0 & . & . & . & 0 \end{pmatrix} = \begin{pmatrix} t_{11} & 0 & . & . & . & 0 \\ t_{21} & 0 & . & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ . & . & . & . & . & . \\ t_{n1} & 0 & . & . & . & 0 \end{pmatrix} (\alpha E_{n1}) \in S(\alpha E_{n1}).$$

Therefore, $l_S(r_S(\alpha E_{n1})) = S(\alpha E_{n1})$.

(2): Let $0 \neq \alpha \in N(R)$ and take, $u = \alpha E_{n1}$. Now, $M_n(R)$ is right Weakly np-injective, so there exists $m \geq 1$ such that $um \neq 0$ and $S(r_S(u)) = Su^m$. Since $n \geq 2$,

$$u^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 0.$$

So, it must be that $m = 1$ and $l_S(r_S((u)) = Su$. Thus, R is right np-injective.

4. NP-INJECTIVE MODULES

We study the main result about np-injective module and short exact sequence. Firstly, we redefine that a module $\boldsymbol{M}$ is called np-injective module if each $\boldsymbol{\alpha} \notin \boldsymbol{N}(\boldsymbol{R})$ and each R-homomorphism $\boldsymbol{f}: \boldsymbol{\alpha R} \to \boldsymbol{M}$, there exists a R-homomorphism $\boldsymbol{g}: \boldsymbol{R} \to \boldsymbol{M}$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})$, for every $\boldsymbol{x} \in \boldsymbol{\alpha R}$, or equivalently, the Figure 4.1 is commutative.

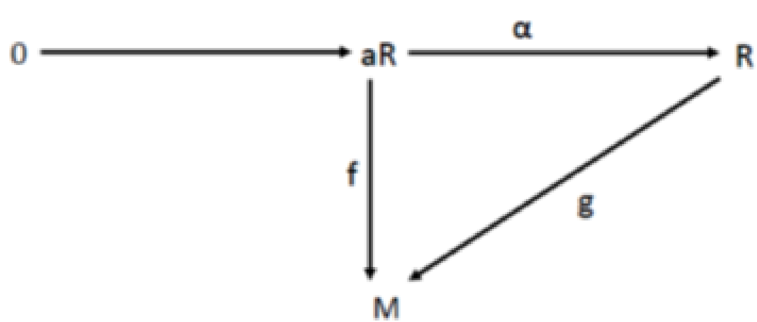

Figure 4.1 : Graph of R−Homomorphism Commutative

Where $\boldsymbol{\alpha}$ is injective R-homomorphism. Furthermore, let $\boldsymbol{M}$ and $\boldsymbol{N}$ be two R-modules. So, the set of all the abelian group of R-homomorphism from $\boldsymbol{M}$ to $\boldsymbol{N}$ are denoted by $\boldsymbol{Hom_R}(\boldsymbol{M},\boldsymbol{N})$, given by $(\boldsymbol{f} + \boldsymbol{g})(\boldsymbol{m}) = \boldsymbol{f}(\boldsymbol{m}) + \boldsymbol{g}(\boldsymbol{m})$ for every $\boldsymbol{m} \in \boldsymbol{M}$ and $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{Hom_R}(\boldsymbol{M},\boldsymbol{N})$. The main result of this section is equivalent of np-injective modules in terms of short exact sequence of modules[6].

Theorem 4.1 [Theorem 1.1., [10]] Let $f: M \to N$ be an R-homomorphism, let A be a submodule of M, let B be a submodule of N, and let $f(A) \subseteq B$. Define the function $\rho: M/A \to N/B$ by $\rho(m+A) = f(m) + B$. We say that ρ is the R-homomorphism induced by f. Then ρ is an R-epimorphism if f is an R-epimorphism, and ρ is a R-monomorphism if $A = f^{-1}(B)$.

Corollary 4.2 [Corollary 1.1.1, [10]] Let $f: M \to N$ be an R-epimorphism. Then, $M/Ker(f) \cong N$.

Proposition 4.3 If M_R is np-injective and B_R is weakly np-injective, then $M \oplus B$ is weakly np-injective.

Proof. Let $0 \neq a \in N(R)$. As, B is a weakly np-injective, then there exists $n \in N$ such that $a^n \neq 0$ and every left R-homomorphism $a^n R \to B$ extends to a homomorphism from R to B. Suppose, $f: a^n R \to M \oplus B$ is a right R-homomorphism. Let $k_1: M \oplus B \to M$ and $k_2: M \oplus B \to B$ be the projections. Now, the right R-homomorphism $f_{k_2} a^n R \to B$ extends to $R \to B$. So, there exists $b \in B$ such that $f_{k_2}(a^n t) = ba^n t$, for each $a^n t \in a^n R$. Since M is np-injective the right R-homomorphism $f_{k_1}: a^n R \to B$ extends to a homomorphism from R to M, So there exists $m \in M$ such that $f_{k_1}(a^n t) = ma^n t$, for each $a^n t \in a^n R$. Now, for each $(m, b) \in M \oplus B$ and for each $a^n t \in a^n R$ we have $(a^n t) = \left(f_{k_1}(a^n t), f_{k_2}(a^n t)\right) = (ma^n t, ba^n t) = (m, b)a^n t$. Thus, $g: a^n R \to M \oplus B$ extends to $R \to M \oplus B$. Hence, $M \oplus B$ is weakly np-injective.

Theorem 4.4 Assume M is an R-module. Then, M is np-injective if and only if for any $a \notin N(R)$, short exact sequence $0 \to aR \xrightarrow{\lambda} R \xrightarrow{\delta} R/aR \to 0$ of R-modules

$$0 \to Hom_R(R/aR, M) \xrightarrow{\mathcal{L}} Hom_R(R, M) \xrightarrow{\mathcal{F}} Hom_R(aR, M) \to 0$$

is also a short exact sequence, where $\mathcal{F}(f) = f\delta$ and $\mathcal{L}(f) = f\lambda$.

Proof. Assume M is a np-injective module. We shall first demonstrate that $\mathcal{F}$ is an R-monomorphism. Assume $\mathcal{F}(f) = \mathcal{F}(g)$, for some $f, g \in Hom_R(R/aR, M)$. Then, for every $b \in R$, $f(\mathcal{F}(b)) = g(\mathcal{F}(b))$. We want to prove that $f = g$. Since δ is an R-epimorphism, then for each $(c + aR) \in R/aR$, there exists $b \in R$ such that $\delta(b) = c + aR$. Then, for each $(c + aR) \in R/aR, f(c + aR) = f(\delta(b)) = g(\delta(b)) = g(c)$. Thus, $f = g$. Consequently, $\mathcal{F}$ is an R-monomorphism. Second, we discover that $\mathcal{L}$ is an R-epimorphism. Let $g \in Hom_R(aR, M)$. We know that M is np-injective, there exists $f \in Hom_R(R, M)$ such that $g = f\lambda = \mathcal{L}(f)$. Therefore, $\mathcal{L}$ is an R-epimorphism. Finally, we demonstrate that $Ker(\mathcal{L}) = Im(\mathcal{F})$. Let $f \in Im(\mathcal{F})$, so there exists $g \in Hom_R(R/aR, M)$ such that $\mathcal{L}(g) = g\lambda = f$. Let $x \in aR$. So, $\lambda(x) \in Im(\lambda) = Ker(\delta)$. Then $\delta(\lambda(x)) = 0$, so $\mathcal{F}\left(g\big(\lambda(x)\big)\right) = g\delta\big(\lambda(x)\big) = g(0 + aR) = 0$. Since $\mathcal{F}$ is an R-monomorphism, then $g\big(\lambda(x)\big) = f(x) = \mathcal{L}(f) = 0$. Therefore, $f \in Ker(\mathcal{L})$, so that $Im(\mathcal{L}) \subseteq Ker(\mathcal{L})$. Let $f \in Ker(\mathcal{L})$ be an arbitrary. Then, $\mathcal{L}(f) = f\lambda = 0$, so that $0 = f(Im\lambda) = f(Ker\delta)$. We know $Ker(\delta) \subseteq R$ and $f(Ker\delta) \subseteq 0$, there is an induced R-homomorphism $\rho: R/Ker(\delta) \to M$ given by $\rho(m + Ker\delta)) = f(m)$ by Theorem 4.1. Also, since $Ker(\delta) \subseteq R$ and $\delta(Ker\delta) \subseteq 0$, there is an induced R-isomorphism $\varepsilon: R/Ker\delta \to R/aR$ which is given by $\varepsilon(b + Ker\delta) = \delta(b)$ by Corollary 4.2. Consider $\rho\varepsilon^{-1}: R/aR \to M$. Notice that $\rho\varepsilon^{-1}$ is an R-homomorphism since ρ and ε^{-1} are R-homomorphism. Since $\varphi(b + Ker\delta) = \psi(b)$ implies that $b + Ker\delta = \varepsilon^{-1}\delta(b)$, for every $b \in R$, $\rho\varepsilon^{-1}\delta(b) = \rho(b + Ker\delta) = f(b)$, so $f = \rho\varepsilon^{-1}\delta = \mathcal{F}(\rho\varepsilon^{-1})$. Thus, $f \in Im(\mathcal{F})$ and $Ker(\mathcal{L}) \subseteq Im(\mathcal{F})$. Therefore, $Ker(\mathcal{L}) = Im(\mathcal{F})$ and hence,

$0 \rightarrow Hom_R(R/aR, M) \xrightarrow{\mathcal{L}} Hom_R(R, M) \xrightarrow{\mathcal{F}} Hom_R(aR, M) \rightarrow 0$ is a short exact sequence. Assume that whenever $0 \rightarrow aR \rightarrow R \rightarrow R/aR \rightarrow 0$ is an exact sequence between R-modules, $0 \rightarrow Hom_R(R/aR, M) \rightarrow Hom_R(R, M) \rightarrow Hom_R(aR, M) \rightarrow 0$ is also exact. Since aR is a submodule of an R-module R, then $0 \rightarrow aR \xrightarrow{\iota} R$ is a sequence with the inclusion map ι and $f: aR \rightarrow M$ is an R-homomorphism. As we see Figure 4.2, $\pi: R \rightarrow R/aR$ is the projection R-homomorphism. We know that the inclusion map ι is an R-monomorphism, the projection map π is an R-epimorphism, and $Ker(\pi) = Im(\iota)$, the above row is exact. Then,

$0 \rightarrow Hom_R(R/aR, M) \xrightarrow{\Pi} Hom_R(R, M) \xrightarrow{I} Hom_R(aR, M) \rightarrow 0$ is also exact. Since $f \in Hom_R(aR, M)$ and aR is an R-epimorphism, there exists g $\in Hom_R(R, M)$ such that $I(g) = g\iota = g|I = f$. As a result, M is np-injective.

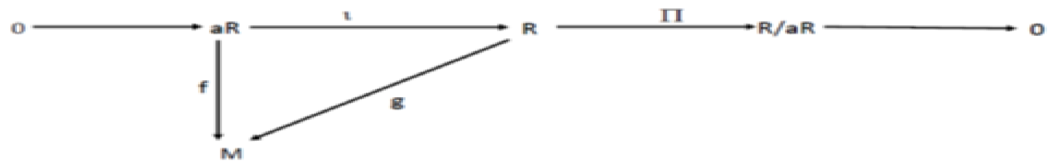

Figure4.2: Exact Sequence between R-Modules

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