Regular Restrained Domination in Middle Graph

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Abstract: In this paper, we introduce the new concept called regular restrained domination in middle graph.A set S ⊆ V[M(G)] is a restrained dominating set if every vertex in V-S is adjacent to a vertex in S and another vertex in V-S. Note that every graph has a restrained dominating set, since S=V is such a set. Let **γ_{rr}[M(G)]** denote the size of a smallest restrained dominating set. Also we study the graph theoretic properties of **γ_{rr}[M(G)]** and many bounds were obtained in terms of elements of G and its relationships with other domination parameters were found.

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1. INTRODUCTION

In this paper, we follow the notations of [4]. All graphs considered here are simple and finite. As usual p= |V| and q= |E| denote the number of vertices and edges of a graph respectively.

In general, we use <X> to denote the subgraph induced by the set of vertices X and N(v) (N[v]) denote the open (closed) neighbourhoods of a vertex v.

The notation $\propto_0(G)(\propto_1(G))$ is the minimum number of vertices (edges) in a vertex (edge) cover of G. The notation $\beta_0(G)(\beta_1(G))$ is the minimum number of vertices (edges) in a maximal independent set of a vertex (edge) of G. Let deg (v) is the degree of a vertex v and as usual $\delta(G)(\Delta(G))$ is the minimum (maximum) degree.

A middle graph M(G) of a graph G is the graph in which the vertex set is V(G)UE(G) and two vertices are adjacent if and only if either they are adjacent edges of G or one is vertex of G and the other is an edge incident with it.

We begin by calling some standard definitions from domination theory.

A set S⊆V(G) is said to be a dominating set of G, if every vertex in V-S is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by **γ**(G)[6].

A dominating set S is called the total dominating set, if for every vertex v∈V, there exists a vertex u∈S, u≠v such that u is adjacent to v. The total domination number of G, denoted by **γ_t** is the minimum cardinality of a total dominating set of G. This is due to E.J.Cockayne, R.M.Dawes and S.T.Hedetniemi [1].

In [7], a connected dominating set D to be a dominating set D whose induced subgraph <D> is connected. The connected domination number $\gamma_c(G)$ of a connected graph G is the minimum cardinality of a connected dominating set.

A dominating set D of a graph G=(V,E) is a split dominating set if the induced subgraph <(V-D)> is disconnected. The split domination number $\gamma_s(G)$ of a graph G is the minimum cardinality of a split dominating set developed by Kulli [8].

A dominating set D of a graph G is a cototal dominating set if the induced subgraph <V-D> has no isolated vertices. The cototal domination number $\gamma_{cot}(G)$ of a graph G is the minimum cardinality of a cototal dominating set. See [8].

In this paper, we study the graph theoretic properties of $\gamma_{rt}[M(G)]$ and many bounds were obtained in terms of elements of G. Also relationships with other domination parameters were found.

The concept of Roman domination was introduced by, E. J . Cockayne,

E.J. Dreyer Jr, S.M. Hedetniemi in [2].

A Roman dominating function on a graph G(V,E) is a function f: V→{0,1,2} satisfying the condition that every vertex u for which f(u)=0 is adjacent to at least one vertex v for which f(v)=2. The weight of a Roman dominating function is the value

$$f(V) = \sum_{u \in V} f(u)$$

The minimum weight of a Roman dominating function on a graph

G is called the Roman domination number of G.

In [3], defined the restrained domination number such that a dominating set D is said to be a restrained dominating set if every vertex of V-D is adjacent to a vertex of D and adjacent to a vertex of V-D.

RESULTS:

Now in the following theorem we established the relationship between our concept with strong split domination and domination number.

Theorem 1: For any connected (p,q) graph G, $\gamma_{rr}[M(G)] + 2 \geq \gamma_{ss}(G) + \gamma(G)$.

Proof: Let $A=\{u_1,u_2,u_3, ...u_n\} \subseteq V(G)$ such that every vertex of V(G)-A is adjacent to at least one vertex of A and N[A]=V(G). If the induced subgraph <A> is totally disconnected, then A is a γ_{ss} – set of G.

Let $V_1= \{v_1,v_2,v_3, ...v_n\} \subseteq V(G)$ be the set of all nonend vertices in G. Suppose there exists a minimal set of vertices $S = \{v_1,v_2,v_3, ...v_k\} \subseteq V_1$ such that N[S] = V(G). Then S forms a minimal dominating set of G.

Further, let $B = \{ u_1,u_2,u_3, ...u_n\} \subseteq V[M(G)]$ be the set of all end vertices of M(G). Now suppose $A_1 \subseteq A$ and every vertex of $V[M(G)] – \{ A_1 \cup B\}$ is adjacent with at least one vertex of $\{ A_1 \cup B\}$ and at least one vertex of $V[M(G)] – \{ A_1 \cup B\}$ such that $N[A_1 \cup B] = V[M(G)]$, which gives $\{ A_1 \cup B\}$ is a restrained dominating set of M(G). If the induced graph of $< A_1 \cup B >$ is regular then $\{A_1 \cup B\}$ is a γ_{rr} – set of M(G). It follows that $| \{A_1 \cup B\} | + 2 \geq |A| + |S|$. Hence $\gamma_{rr}[M(G)] + 2 \geq \gamma_{ss}(G) + \gamma(G)$.

Theorem 2: For any connected (p,q) graph G, $p \geq \gamma_{rr}[M(G)]$.

Proof: Let $C= \{ u_1,u_2,u_3, ...u_n\} \subseteq V[M(G)]$ be the set of all end vertices of M(G). Now suppose $A_1 \subset V[M(G)]$ such that $N[A_1 \cup C] = V[M(G)]$. Also $\forall v_i \in V[M(G)] – \{ A_1 \cup C\}$ is adjacent to at least one vertex of $V[M(G)] – \{ A_1 \cup C\}$ and at least one vertex of $\{ A_1 \cup C\}$. Then clearly $\{ A_1 \cup C\}$ is a restrained dominating set of M(G). Suppose the induced subgraph $< A_1 \cup C >$ is regular. Then $\{ A_1 \cup C\}$ is a γ_{rr} - set (G). Since |p| = V(G). It follows that $|p| \geq |\{ A_1 \cup C \}|$. Hence $p \geq \gamma_{rr}[M(G)]$.

Theorem 3: For any connected (p,q) graph G, $q \leq \gamma_{rr}[M(G)] + \beta_1(G) + 2$.

Proof: Let W= { $e_1,e_2,e_3, ...e_n$} = E(G). Suppose W_1 = { $e_1,e_2,e_3, ...e_m$}⊆E(G) be the maximal set of edges with $N(e_i) \cap N(e_j)$ = e and e∊ W-W_1. Clearly, W_1 forms a maximal independent edge set in G.

Further, since V[M(G)] = V(G) U E(G). Let D = { $u_1,u_2,u_3, ...u_n$}⊆V[M(G)] be the γ – set of M(G). Suppose there exists a set A = {$v_1,v_2,v_3, ...v_m$} ⊆ V[M(G)], such that ∀v_i∊A, 1≤ i ≤ m are the vertices with maximum degree. Let D ⊂ V[M(G)] be the set of all end vertices and N[D_1 U D] = V[M(G)]. Clearly { D_1 U D} is a dominating set of M(G). Suppose ∀v_i∊ V(G) – { D_1 U D} is adjacent to at least one vertex of { D_1 U D} and V[M(G)] – { D_1 U D}. If the induced subgraph of <D_1 U D> is regular, then clearly D_1 U D is a γ_{rr} – set of M(G). Since q = E(G). It follows that |q| ≤ | D_1 U D | + |W_1|+ 2, which gives $q \leq \gamma_{rr}[M(G)] + \beta_1(G) + 2$.

Theorem 4: For any connected (p,q) graph G, $\gamma_c(G) + \alpha_1(G) \geq \gamma_{rr}[M(G)]$.

Proof: Let B={$v_1,v_2,v_3, ...v_n$}⊆V(G) be the minimal set of vertices which covers all the vertices of G such that N[B]=V(G). Then B is a γ – set of G. Further, if the induced subgraph <B> has exactly one component, then B itself is a connected dominating set of G. Otherwise if B has more than one component, then attach minimum set of vertices {w_i} from V(G) – B which are in u – w path ∀ u, v ∊ V-B gives a single component B_1=BU{w_i}. Clearly B_1 forms a minimal γ_c – set of G.

Suppose A= { $e_1,e_2,e_3, ...e_m$}⊆E(G) be the maximal set of edges with $N(e_i) \cap N(e_j)$=e, ∀ e_i, e_j ∊ B, 1≤i≤n, 1≤j≤n and e ∊ E(G) – A. Clearly A forms a maximal independent edge set in G. Suppose K= {$v_1,v_2,v_3, ...v_n$} be the set of vertices which are incident with the edges of A and if |K|=p, then K itself is an edge covering number. Otherwise consider the minimum number of edges {e_m}⊆ E(G) – K, such that A_1=KU{e_m} forms a minimal edge covering set of G.

Further, let X={ $u_1,u_2,u_3, ...u_n$} ⊆ V[M(G)] be the set of all nonend vertices of M(G). Now suppose X_1⊆ B and every vertex of V[M(G)]- {X_1UX} is adjacent with at least one vertex of {X_1UX} and at least one vertex of V[M(G)]- {X_1UX} such that N[X_1UX]=V[M(G)], which gives {X_1UX} is a restrained dominating set of M(G). If the induced subgraph < X_1UX > is regular, then {X_1UX} is a γ_{rr}[M(G)]. Hence |B_1|+|A_1| ≥ |{X_1UX}| which gives $\gamma_c(G) + \alpha_1(G) \geq \gamma_{rr}[M(G)]$.

Theorem 5: For any connected (p,q) graph G, $\gamma_R(G) + \Delta(G) \geq \gamma_{rr}[M(G)]$.

Proof: Let f : V(G) → {0,1,2} and partition the vertex set of V(G) into [V_0,V_1,V_2] induced by f with $|V_i| = n_i$ for i=0,1,2. Suppose the set V_2 dominates V_0, then S=V_1UV_2 forms a minimal roman dominating set of G.

Further, since V[M(G)] = V(G)UE(G). Suppose there exists K ⊆ V[M(G)] and N[K]=V[M(G)]. Then K is a minimal dominating set of M(G). If for every v_i ∊ {V[M(G)]-K} is adjacent to at least one vertex of K and at least one vertex of {V[M(G)]-K}, then K is a minimal restrained dominating set of M(G). Assume the induced subgraph < K> is regular. Then K is a regular minimal restrained dominating set of M(G). Since for any graph G, then there exists at least one vertex of maximum degree v ∊ V[G], such that deg(v) = Δ(G). Hence |S| + Δ(G) ≥ |K|, which gives $\gamma_R(G) + \Delta(G) \geq \gamma_{rr}[M(G)]$.

Theorem 6: For any connected (p,q) graph G, $\gamma_s(G) + \beta_0(G) + \delta(G) \geq \gamma_{rr}[M(G)]$.

Proof: Let A= {$v_1,v_2,v_3, ...v_n$}⊆ V(G) be the set of all end vertices in G and A' = V(G) – A. Suppose there exists a vertex set D ⊂ A' such that N[D]=V(G). If the induced subgraph <D> has more than one component then D forms a γ_s – set of G.

Let K= { u_1, u_2, u_3, ...u_n} ⊆ V(G) be the minimum set of vertices such that dist(u,v)≥2 and N(u)∩N(v)=x, ∀ u,v ∊ K and x ∊ V(G) – K. Clearly |K|= β_0(G).

Since V[M(G)]=V(G)UE(G). Further, let B= {v_1, v_2, v_3, …..v_n}⊆ V[M(G)] be the set of all end vertices in M(G) and B' = V[M(G)] – B. Then there exists vertex set H ⊆ B' such that N[HUB]=V[M(G)]. So that {HUB} is a dominating set of V[M(G)]. Since ∀ v_i ∊ [M(G) – {HUB}] is adjacent to at least one vertex of {HUB} and V[M(G)] – {HUB} and the induced subgraph < {HUB} > is regular, then {HUB} is a γ_{rr} - set of M(G). For any graph G, there exists one vertex of minimum degree v ∊ V(G), such that deg(v)= δ(G). Since D⊂ V[M(G)] and K ⊂ V[M(G)], then it follows that |D| + |K| + δ(G) ≥ |{HUB}|. Hence γ_s(G) + β_0(G) + δ(G) ≥ γ_{rr}[M(G)].

Theorem 7: For any connected (p,q) graph G, γ_{cot}(G) + diam(G) ≥ γ_{rr}[M(G)].

Proof: Let W={v_1, v_2, v_3, …..v_n}⊆ V(G) be the minimal set of vertices which covers all the vertices of G such that N[W]=V(G). Further if the induced subgraph <V(G) - W> has no isolates, then W is a cototal dominating set of G. Otherwise there exists a set H of vertices which are isolates in <V(G) - W> such that {WUH} forms a minimal total dominating set of G. Clearly {WUH} is a minimal cototal dominating set of G.

Let B ⊆ V(G) be the minimal set of vertices. Further, there exists an edge set J⊆J', where J' is the set of edges which are incident with the vertices of B constituting the longest path in G such that |J|=diam(G).

Further, let K= {v_1, v_2, v_3, …..v_n}⊂ V[M(G)] be the set of all end vertices in M(G) and K_1=V[M(G)]-K. Then there exists a vertex set L⊆ K_1 such that ∀ v_i ∊ V[M(G)]-{LUK} is adjacent to at least one vertex of {LUK} and in [V[M(G)] – {LUK}]. Then {LUK} is a γ_{rr} – set of M(G).

It follows that, |WUH| + |J| ≥ |LUK|. Hence γ_{cot}(G) + diam(G) ≥ γ_{rr}[M(G)].

In [5], given two adjacent vertices u and v we say that u weakly dominates v if deg(u)≤deg(v). A set D⊆V(G) is a weak dominating set of G if every vertex in V-D is weakly dominated by atleast one vertex in D. The weak domination number γ_w(G) is the minimum cardinality of a weak dominating set.

Theorem 8: For any connected (p,q) graph G, γ_w(G) + α_0(G) ≥ γ_{rr}[M(G)].

Proof: Let A= {v_1, v_2, v_3, …..v_n}⊆ V(G) be the minimal dominating set of G. If every vertex u ∊ V(G)-A is adjacent with v∊A and deg(v)≤deg(u). Then A is a weak dominating set of G.

Suppose B={u_1, u_2, u_3, ...u_n}⊆V(G), ∀ e_i ∊ E(G) is incident to at least one vertex of B. Then |B|=α_0(G).

Further, let K={ u_1, u_2, u_3, ...u_n}⊆V[M(G)] be the set of all end vertices in M(G) and K'=V[M(G)]-K. Then there exists vertex set H⊆K' such that N[HUK]=V[M(G)] so that {HUK} is a dominating set of V[M(G)]. Since ∀ v_i ∊ [M(G)-{HUK}] is adjacent to at least one vertex of {HUK} and V[M(G)]-{HUK}. If the induced subgraph <HUK> is regular, then {HUK} is a γ_{rr} - set M(G). Since A⊂ V[M(G)] and B⊂ V[M(G)], then it follows that |A| + |B| ≥ |{HUB}|. Which gives γ_w(G) + α_0(G) ≥ γ_{rr}[M(G)].

Theorem 9: For any connected (p,q) graph G, γ_{rr}[M(G)] + 1 ≥ γ_t(G).

Proof: Let A= {v_1, v_2, v_3, …..v_n}⊆ V(G) be the minimal set of vertices which covers all the vertices in G. Clearly A forms a dominating set of G. Suppose the subgraph <A> has no isolates. Then A itself is a γ_t –set of G. Otherwise if deg(v_k)<1 then attach the vertices w_i ∊ N(v_k) to make deg(v_k)≥1 such that <AU{w_i}> does not contain any isolated vertex. Clearly AU{w_i} forms a total dominating set of G.

Further let B={v_1, v_2, v_3, …..v_k}⊆ V[M(G)] be the set of all end vertices in M(G) and B_1=V[M(G)]-B. Then there exists vertex set H⊆ B_1 such that ∀ v_i ∊ V[M(G)]-{HUB} is adjacent to at least one vertex of {HUV} and a vertex of V[M(G)]-

{HUB}. Then {HUB} is a γ_r – set of M(G). If the induced subgraph <HUB> is regular, then {HUB} is a γ_{rr} – set of M(G). One can easily see that [AU{w_i}]=V[M(G)]. It follows that |HUB| +1 ≥ |AU{w_i}|. Hence $\gamma_{rr}[M(G)] + 1 \geq \gamma_t(G)$.

A set of edges in a graph G=(V,E) is called an edge dominating set of G if every edge in E-F is adjacent to at least one edge in F. Equivalently, a set F of edges in G is called an edge dominating set of G if for every edge e∊F, there exists an edge e_1∊F such that e and e_1 have a vertex in common. The edge domination number γ'(G) of a graph G is the minimum cardinality of edge dominating set of G[9].

<u>Theorem 10:</u> For any connected (p,q) graph G, $\gamma'(G) + \gamma_{st}(G) \leq \gamma_{rr}[M(G)] + 2$.

<u>Proof:</u> Let A={ $e_1, e_2, e_3, \ldots e_n$}⊆E(G), if for every edge e ∊ E-A then there exists an edge e' ∊ A such that e and e' have a common vertex. Then A is a minimal edge dominating set of G.

Let B={$v_1, v_2, v_3, \ldots v_n$} be the vertex set of G. Suppose $B_1 \subseteq B$ such that $N[B_1]=V(G)$. If deg(u)≥deg(v), ∀ u ∊ B_1 and ∀ v ∊ {B- B_1}, u is adjacent to v. Then B_1 is a strong dominating set of G.

Further, let C= { $u_1, u_2, u_3, \ldots u_n$}⊆V[M(G)] be the set of all end vertices in M(G) and C_1=V[M(G)]-C. Then there exists a vertex set H⊆C_1 such that ∀ v_i ∊ V[M(G)]-{HUC} is adjacent to at least one vertex of {HUC} and V[M(G)]-{HUC}. If the induced subgraph <HUC> is regular then {HUC} is γ_{rr} – set of M(G). It follows that |A| + |B_1| ≤ |{HUC}| + 2.

Hence $\gamma'(G) + \gamma_{st}(G) \leq \gamma_{rr}[M(G)] + 2$.

<u>Theorem 11:</u> For any connected (p,q) graph G, $\gamma_{rr}[M(G)] + \gamma(G) \geq \gamma_{st}(G) + \alpha_1(G)$.

<u>Proof:</u> Suppose C={ $e_1, e_2, e_3, \ldots e_n$}⊆E(G) be the minimal set of edges with $N[e_i] \cap N[e_j]=e$, ∀ e_i, e_j ∊ B, 1≤i≤n, 1≤j≤n and e ∊ E(G)-C. Suppose D={$v_1, v_2, v_3, \ldots v_n$} be the set of vertices which are incident with the edges of C and if |D|=P, then D itself is an edge covering number of G. Since γ(G) ⊆V[M(G)], then γ[M(G)] ⊆ γ_{rr}[M(G)] and then from Theorem 10 the result follows.

REFERENCE

[1]. E.J. Cockayne, R.M. Dawes and S.T. Hedetneimi, Total domination in graphs, Networks 10: 211-219 (1980).
[2]. E.J. Cockayne. E.J.P.A. Dreyer Jr and S.M. Hedetniemi, Roman domination in graphs, Discrete Mathematics 278: 11-22 (2004).
[3]. G.S. Domke, J.H. Hattingh, S.T. Hedetneimi, R.C. Laskar and L.R. Markus, Restrained domination in graphs, Discrete Mathematics, 203: 61-69 (1999).
[4]. F. Harary, Graph Theory, Addison-Wesley, Reading Mass (1972).
[5]. J.H. Hattingh and R.C. Laskar, On weak domination in graphs, Ars Combinatorics, 49: 205-216 (1998).
[6]. T.W. Haynes, S.T. Hedetneimi and P.J. Slater, Fundamentals of domination in graphs, Marcel Dekker Inc., New York 1998.
[7]. S.T. Hedetneimi and R.C. Laskar, Connected domination in graphs, Graph Theory and combinatorics, Ed.B. Bollobas, Academic Press, London, 209-218 (1984).
[8]. V.R. Kulli, Theory of domination in graphs, Vishwa International Publication (2010).
[9]. S.L. Mitchell and S.T. Hedetneimi, Edge domination number in trees, Congr. Number., 19:489-509 (1977).

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