Infinite Order of The Solutions of Higher Order Homogeneous Linear Differential Equations with Entire Coefficients Having Completely Regular Growth Property and Bounded Components Fatou Set

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Abstract: The homogeneous higher order complex linear differential equations (n-thCLDEs) with entire functions is considered in this paper. We investigated some conditions that can be put on the coefficients which guarantee that any nonzero solution of such equations has infinite order. The conditions we stated are the completely regular growth (CRG), the characteristic function of some coefficients is approximately equals to the logarithm of its maximum modulus and the Denjoy's conjecture (DC) property.

Keywords: Linear Differential Equations, Entire Functions, Completely Regular Growth, Order of Growth, Denjoy's Conjecture.

1. INTRODUCTION AND PRELIMINARIES

The Nevanlinna theory related to meromorphic functions is a powerful tool in studying the differential equations in the complex plane. The Nevanlinna's theory was first applicable on the complex differential equations (CDE) as it appears in Yoseida's work in 1932, and from that time (CDE) have been became an active field of study by authors. For more details related to the theory of differential equations in $\mathbb{C}$, see, for example [1]. Study the order of growth of solutions of this equation is one of the main purpose. The reader must be familiar with the fundamental definitions and the results related the Nevanlinna value distribution theory of meromorphic functions such as, $M(r,f), T(r,f)$ and $N(r,f)$ etc., see [15, 29].

In this paper we shall consider the following (n-thCLDE):

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_2(z)f'' + A_1(z)f' + A_0(z)f = 0 \tag{1}$$

where f is unknown, $A_j(z), 0 \le j \le n-1$ entire functions (EFs).

In order to state and prove our main results in this paper, we need some definitions and concepts. These concepts can be given as follows:

The concepts of the order of growth and lower order of growth of a meromorphic function f are given as follows [5, 6]:

$$\rho(f) = \lim_{r\to\infty} sup \frac{log^+ T(r,f)}{log\ r}, \text{(order of growth)}$$

$$\mu(f) = \lim_{r\to\infty} inf \frac{log^+ T(r,f)}{log\ r}, \quad \text{(lower order of growth)}$$

If f is EF, then $T(r,f) = \log^+ M(r,f)$, where $M(r,f) = \max_{|z|=r} |f(z)|$ (maximum modulus).

Let $E \subseteq [0,\infty)$, then the concept of linear measure to E is given [14]:

$$m(E) = \int_E dt$$

while the logarithmic measure of a set $E \subseteq [1,\infty)$ is given by [3, 4]:

$$m_l(E) = \int_E \frac{dt}{t}$$

whiles (lower) resp. (upper) logarithmic densities of $E \subseteq [1, \infty)$ are given as [17]:

$$\underline{\text{logdens}}E = \lim_{r\to\infty} \inf \frac{m_l(E\cap[1,r])}{\log r}$$

and

$$\overline{\text{logdens}}E = \lim_{r\to\infty} \sup \frac{m_l(E\cap[1,r])}{\log r}$$

if

$$\underline{\text{logdens}}E = \overline{\text{logdens}}E$$

then E is said to has logarithmic density

In order to prove our results, we need the following important concepts [9]:

if $\rho(r) > 0$, is differentiable with $\lim_{r\to\infty} \rho(r) = \rho\ (0 < \rho < \infty)$, $\lim_{r\to\infty} \rho'(r)\, r\log r = 0$, then $\rho(r)$ is said a proximate order. Let $g(z)$ be an EF of order ρ, then the indicator $h(\theta)$ of g with respect to $\rho(r)$ is defined by

$$h(\theta) = \lim_{r\to\infty} \sup \frac{\log\left|g(re^{i\theta})\right|}{r^{\rho(r)}}$$

where $\lim_{r\to\infty}(\rho(r) \to \rho)$, then $g(z)$ is said to be (in the sense of Puger and Levin) completely regular growth (CRG) if the disks $D(a_k, s_k)$ exists that satisfy

$$\sum_{|a_k|\leq r} s_k = o(r)$$

such that

$$\log\left|g(re^{i\theta})\right| = h(\theta)r^{\rho(r)} + o\left(r^{\rho(r)}\right), \qquad re^{i\theta} \notin \cup_k D(a_k, s_k) \tag{2}$$

as $r \to \infty$, uniformly in θ. The $\bigcup_k D(a_k, s_k)$ satisfying (2) is called a C_0 set, see [10] for more details of CRG. A results related to the coefficient of CDE with dynamical property can be found in [12].

Definition 1 [2] Let $g(z)$ be an EF satisfying the following:

1. Let $\arg z = \theta_j, j = 1, 2, \ldots, m$ be the accumulated lines zeros of $g(z)$ and $\theta_1 < \theta_2 < \cdots < \theta_m < \theta_{m+1} = \theta_1 + 2\pi$;
2. $h(\theta)$ is an indicator of $g(z)$ in $S(\theta_j, \theta_{j+1}) = \{z = re^{i\theta}: r > 0, \theta_j < \theta < \theta_{j+1}\}, j = 1, 2, \ldots, m$ and $\rho(r)(\to \rho)$ is a proximate order to $g(z)$;
3. $\epsilon(r) = \frac{1}{\log^N r}$ where $N \in \mathbb{N}$ and $\log^N$ denotes the logarithm of $N-\text{th}$ iterate;
4. $\log\left|g(re^{i\theta})\right| = h(\theta)r^{\rho} + O\left(r^{\rho(r)}\epsilon(r)\right)$ for $\left|\theta - \theta_j\right| > \epsilon(r), j = 1;,2;\ldots, m.$

then $g(z)$ is said to has the property of special completely regular growth (SCRG).

Definition 2 [21] Let $g(z)$ be EF with $0 < \mu(g) < 1$. A ray $0 \leq \arg z > 2\pi$ of origin beginning is called a Borel direction of order greater than or equal to $\mu(g)$ of g, if for each $\epsilon > 0$ and for any $a \in \mathbb{C} \cup \{\infty\}$, possibly with two exceptions, the following holds

$$\lim_{r\to\infty} \sup \frac{\log n\left(S(\theta-\epsilon, \theta+\epsilon, r), \alpha, g\right)}{\log r} \geq \mu(g)$$

where $n\left(S(\theta-\epsilon, \theta+\epsilon, r), \alpha, g\right)$ denotes the number of zeros, counting the multiplicities, of $g - a$ in $S(\theta-\epsilon, \theta+\epsilon, r) = \{z: \theta - \epsilon < \arg z < \theta + \epsilon, |z| < r\}$.

The Borel direction concept with order $\rho(g)$ is obtain when $(\geq \mu(g))$ is replaced with $= \rho(g)$.

In what follows, the concepts of EFs that are extremal for DC will be given.

In 1907, Denjoy [24] appeared a conjecture which is: if $g(z)$ be ED with $\rho(g) < \infty$ and $g(z)$ has k asymptotic values that are distinct and finite, then $k \le \rho(g)$. Ahlfors was verified this concept in 1930 [23]. If $k = \rho(g)$ then $g(z)$ is called extremal for DC. This type of functions is well investigated, for instance see ([23], [21]).

The authors of [22] studied the solutions growth of CDE that has coefficients with extremal for DC.

In the below, we shall give a brief introduction related to complex dynamics, see [19].

Let f be a transcendental EF. The Fatou set $\mathcal{F}(f) \subseteq \mathbb{C}$ of f where f^n of f represents a normal family. The Julia set $\mathfrak{J}(f)$ of f is $\mathcal{F}(f)^c$ in $\mathbb{C}$. $\mathcal{F}(f)$ has completely invariant property under f, in other words $z \in \mathcal{F}(f) \Leftrightarrow f(z) \in \mathcal{F}(f)$. Hence, if U is a member of $\mathcal{F}(f)$ (Fatou member), ,then there is a Fatou component U_n for some $n = 0,1,2,\dots$, such that $f^n(U) \subseteq U_n$. If $U_p = U_0 = U$ for some $p \ge 1$, then U is called a periodic member with period p, supposing that p minimal. If U_n is not eventually periodic, then U is a domain of f that wandering. Although the some EFs that have simply connected Fatou member only, such as the class of (Eremenko – Lyubich) function [20], a many types of EF that have multiply connected Fatou members are exists. The first such function of this type is constructed in [15], who proved in [18] that this type of function is with a multiply connected Fatou member property which is a wandering domain. The author of [17] proved this property isn't special case of the example; for if U is any multiply connected Fatou member to f, then U is the wandering domain, it's called (Baker wandering domain), which has properties says: 1) every U_n is multiply connected and bounded; ;2) $\exists N \in \mathbb{N}$ s. t. U_n and 0 belong to a complementary bounded member of $U_{n+1}, n \ge N$; 3) $\lim_{n\to\infty} \operatorname{dis}(U_n, 0) \to 0$. Thus, if f has a connected multiply Fatou member, then $\mathfrak{J}(f)$ with bounded component only.

Our main purpose in this paper is to answer the following question: what conditions should give on the coefficients which guarantee that any nontrivial solution of Eq. ( ) has infinite order ?
There are many authors answer this question but on 2nd order linear CDE:

$$f'' + A(z)f' + B(z)f = 0 \quad (3)$$

These results can be summarized as follows [28]:

Theorem 1 Let $A(z)$ be CRG entire function and the set $E = \{\theta \in [0,2\pi): h(\theta) = 0\}$ has $m(E) = 0$, $B(z)$ be a transcendental EF with a multiply connected Fatou component s. t. $\rho(A) \ne \rho(B)$. Then any nonzero solution of (3) has infinite order.

Theorem 2 Let $A(z)$ be CRG entire function and the set $E = \{\theta \in [0,2\pi): h(\theta) = 0\}$ has $m(E) = 0$, $B(z)$ be a transcendental EF with $T(r,B) \sim \log M(r,B)$ as $r \to \infty$ outside a set which has a finite Lebesgue measure s. t. $\rho(A), \rho(B)$ aren't equal. Then any solution $f \ne 0$ of (3) has infinite order.

Theorem 3 Let $A(z)$ be EF that is CRG and the set $E = \{\theta \in [0,2\pi): h(\theta) = 0\}$ has $m(E) = 0$, $B(z)$ be a transcendental EF that extremal for DC s. t. $\rho(A), \rho(B)$ aren't equal. Then any solution $f \ne 0$ of (5) has infinite order.
There are some results about the coefficients of Eq. (3), such as [13, 11].

We will give the answer the previous question, we will state the conditions on the coefficients that guarantee that all nontrivial solution of (1) is of infinite order.

2. MATERIEL AND METHODS

In this section we will give some results which will be used to prove our main results.

Lemma 1 [7, 8] Let f be transcendental meromorphic function with $\rho(f) < \infty$, $\varepsilon > 0$ and $k > j \geq 0$. Then there is $E \subseteq [1, \infty)$ with $m_l(E) < \infty$, such that the following holds

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c|z|^{(k-j)(\rho(f)-1+\varepsilon)}, \qquad |z| = r \notin E_2 \cup [0,1] \tag{4}$$

Lemma 2 [25] Let $A(z)$, $B(z)(\neq 0)$ be EFs, s. t. for $\alpha, \beta, \theta_1, \theta_2 \in \mathbb{R}$, where $\alpha > 0$; $\beta > 0$ and $\theta_1 < \theta_1$,

$$A(z) \geq \exp\{(1 + o(1))\alpha|z|^{\beta}\}$$

and

$$B(z) \leq \exp\{o(1))|z|^{\beta}\}$$

as, $z \to \infty$ in $\bar{S}(\theta_1, \theta_2) = \{z: \theta_1 \leq \arg z \leq \theta_2\}$. Let $\epsilon > 0$ small enough, and let $\bar{S}(\theta_1 + \epsilon, \theta_2 - \epsilon) = \{z: \theta_1 + \epsilon \leq \arg z \leq \theta_2 - \epsilon\}$. If $f \neq 0$ is a solution of Eq. 3 with $\rho(f) < \infty$, then the following hold:

1. There is $b(\neq 0)$ s. t. $\lim_{z \to \infty} f(z)$ exists in $\bar{S}(\theta_1 + \epsilon, \theta_2 - \epsilon)$. Further,

$$f(z) - b \leq \exp\{-(1 + o(1))\alpha|z|^{\beta}\}$$

as, $z \to \infty$ in $\bar{S}(\theta_1 + \epsilon, \theta_2 - \epsilon)$;

2. $\left|f^{k}(z)\right| \leq \exp\left\{-\left(1 + o(1)\right)\alpha|z|^{\beta}\right\}, k \geq 1$ as $z \to \infty$ in $\bar{S}(\theta_1 + \epsilon, \theta_2 - \epsilon)$.

Lemma 3 [26] Let $\phi(r)$ is continuous, nondecreasing function on $\mathbb{R}^{+}$. Assume

$$\limsup_{r \to \infty} \frac{\log \phi(r)}{\log r} > \alpha > 0$$

define $G = \{r \in \mathbb{R}^{+}: \phi(r) \geq r^{\alpha}\}$. Then $\overline{\text{logdens}}(G) > 0$.

Lemma 4 [21] Let $g(z)$ is EF that extremal for DC. Then, for $0 \leq \theta < 2\pi$, either $\arg z = \theta$ Borel direction with order $\rho(g)$ of $g(z)$ or $\exists \delta, 0 < \delta < \frac{\pi}{4}$, s. t.

$$\lim_{r \to \infty} \frac{\log \log|g(z)|}{\log r} = \rho(g)$$

for all $z \in S(\theta - \delta, \theta + \delta) \backslash E$, $E \subseteq S(\theta - \delta, \theta + \delta)$, and the following satisfy

$$\lim m(S(\theta - \delta, \theta + \delta;\ r, \infty) \cap E) = 0$$

where, $S(\theta - \delta, \theta + \delta;\ r,\ \infty) = \{z: \theta - \delta \leq \arg z \leq \theta + \delta,\ 0 \leq |r| < \infty\}$.

Lemma 5 [16] Let $f(z)$ is a function transcendental meromorphic having finite poles. If $\mathfrak{I}(f)$ has bounded components only, then to each $a \in \mathbb{C}$, $\exists d, 0 < d < 1$ and 2 sequences $\{r_n\}$, $\{R_n\}$ $r_n > 0, R_n > 0$ with $r_n \to \infty$ and $\frac{R_n}{r_n} \to \infty$ $(n \to \infty)$ s. t.

$$M_c(r, a, f)^{d} \leq L_c(r, a, f), \ \ r \in G = \cup_{n=1}^{\infty} \{r: r_n < r < R_n\}$$

Lemma 6 [27] Let $g(z)$ be EF of order $\rho(g) \in (0, \infty)$, and let $S(\varphi_1, \varphi_2) = \{z: \varphi_1 < \arg z < \varphi_2\}$ be a sector with $\varphi_2 - \varphi_1 < \frac{\pi}{\rho(g)}$. If there exists a Borel direction of order $\rho(g)$ of $g(z)$ in $S(\varphi_1, \varphi_2)$, then for at least one of the two rays $L_j = \{z: \arg z = \varphi_j, j = 1,2\}$, say L_2, we have

$$\limsup_{r \to \infty} \frac{\log \log\left|g(re^{i\varphi_2})\right|}{\log r} = \rho(g) \tag{5}$$

3. RESULTS AND DISCUSSIONS

In this section we will generalize the previous results related to Eq. (3) to the (n-thCLDE) in which the conditions on the coefficients are modified and the condition of the difference between the order of growth of the coefficients is omitted.

Theorem 4 Let $A_j(z)$ be EFs that are CRG and $E = \{\theta \in [0,2\pi): h(\theta) = 0\}$ be with $m(E) = 0$, and let $A_0(z)$ be a transcendental EF where $\mathfrak{I}(A_0)$ has only bounded components. Then every solution $f \neq 0$ of (1) has infinite order.

Proof Assume $f \neq 0$ where $\rho(f) < \infty$. Put $E^* = \{\theta \in [0,2\pi): h(\theta) \leq 0\}$. We have 2 cases depending on $m(E^*) > 0$ or $m(E^*) = 0$.

Case 1. Suppose $m(E^*) > 0$, then there is a sectors such that the $A_s(z)$'s indicator, $1 \leq s \leq n-1$ is negative. A ray $\arg z = \theta^*$ in these sectors can be chosen so that $h(\theta^*) < 0$. By Lemma 1, there is $E \subseteq (1,\infty)$ with $m(E) < \infty$ such that for all $|z| = r \notin EU\,[0,1]$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |z|^{2\rho(f)}, k = 1, 2, \dots, n \tag{6}$$

Using Lemma 5, there is $z_n = r_n e^{i\theta^*}$ with $r_n \in G\backslash EU\,[0,1]$ with $\lim_{n\to\infty} r_n = \infty$ such that

$$M(r_n, A_0) < L(r_n, A_0) \leq |A_0(z)| \leq \left|\frac{f^{(n)}(z_n)}{f(z_n)}\right| + \sum_{j=1}^{n-1}\left|\frac{f^{(j)}(z_n)}{f(z_n)}\right| A_j(z_n) + \dots + \left|\frac{f'(z_n)}{f(z_n)}\right| \leq (n-1)(1+o(1))r_n^{2\rho(f)}$$

where $(0 < d < 1)$. Because $A_0(z)$ is a nonconstant EF, then we have

$$dT(r_n, A_0) < d\log M(r_n, A_0) \leq 2\rho(f)\log r_n + o(1) \tag{7}$$

as $\lim_{n\to\infty} r_n = \infty$. Since $A_0(z)$ is transcendental, we have $\lim_{r_n\to\infty} \frac{T(r_n,A_0)}{\log r_n} = \infty$. Therefore we have a contradiction from (7).

Case 2. Suppose that $m(E^*) = 0$, then the indicator of $A_s(z), 1 \leq s \leq n-1$ satisfies $h(\theta) > 0, \theta \in [0,2\pi)\backslash E^*$. By part (4) of definition 1, we have

$$\log A_s(re^{i\theta}) = h(\theta)r^{\rho(r)} + o(r^{\rho(r)})$$

for $z = re^{i\theta}$ with $\theta \in [0,2\pi)\backslash E^*$ $z \notin C_0$ set, and $\lim_{r\to\infty} \rho(r) \to \rho(A_s)$. Then for any given $(0 < \delta < \frac{\pi}{4\rho})$ and $(0 < \eta < \frac{\rho-\rho(A_0)}{4})$, we have

$$|A_j(z)| \geq \exp\left\{(1+o(1))\alpha|z|^{\rho(A_j)-\eta}\right\};$$

$$|A_0(z)| \leq \exp\{|z|^{\rho(A_0)+\eta}\} \leq \exp\left\{|z|^{\rho(A_j)-2\eta}\right\} \leq \exp\left\{o(1)|z|^{\rho(A_j)-\eta}\right\};$$

for $1 \leq j \leq n-1$, where $z = re^{i\theta}$ and θ as specified above, $\alpha(\delta) > 0$. Then by Lemma 2, there is a constant $b_j \neq 0$ s. t.

$$|f(z) - b_j| \leq \exp\{-(1+o(1))\alpha|z|^{\rho-\eta}\};$$

where $z = re^{i\theta}$ and θ as specified above. Then using principle due to Phragmen-Lindelof, $f(z)$ is bounded in $\mathbb{C}$. Thus using Liouville's theorem we conclude that f is a constant in $\mathbb{C}$, which is a contradiction. Therefore the proof is completed.

Theorem 5 Let $A_j(z), 1 \leq j \leq n-1$ be EFs that are CRG and $E = \{\theta \in [0,2\pi): h(\theta) = 0\}$ is with $m(E) = 0$, $A_0(z)$ is a transcendental EF with $T(r, A_0) \sim \log M(r, A_0)$, as $r \to \infty$ for some G with $m(G) < \infty$ and $z \notin G$. Then any nonzero solution of (1) has infinite order.

Proof let $f \neq 0$ be a solution of (1) with $\rho(f) < \infty$. We test two cases based on $m(E^*) = 0$ or $m(E^*) > 0$.

Case 1. Suppose $m(E^*) = 0$, then $h(\theta)$ of $A_s(z), 1 \leq s \leq n-1$ is positive for $\theta \in [0, 2\pi) \setminus E^*$. The conclusion is similar to Case. 2 in the previous theorem.

Case 2. Suppose $m(E) > 0$, then there is a sectors in which $h(\theta)$ of $A_s(z), 1 \leq s \leq n-1$ is negative. Thus, there is $I_{A_s} \subseteq [0, 2\pi)$ where I_{A_s} is interval s. t. $h(\theta), \theta \in I_{A_s}$ is negative. By Lemma 1, there is $E_1 \subseteq (1, \infty)$ having $m(E_1) < \infty$ such that z with $|z| = r \notin E_1 \cup [\,[0, 1]$, (12) holds. Set

$$I_{A_0}(r) = \{\theta \in [0, 2\pi): \log|A_0(re^{i\theta})| \leq c\log M(r, A_0), 0 < c < 1\} \tag{8}$$

and let $m(I_{A_0}(r))$ be its Lebesgue measure. From proximate function definition $m(r; A_0)$ it follows

$$T(r, A_0) = m(r, A_0) \leq \left(\frac{2\pi - m\left(I_{A_0}(r)\right)}{2\pi}\right)\log M(r, A_0) + c\frac{m\left(I_{A_0}(r)\right)}{2\pi}\log M(r, A_0)$$

Therefore, $T(r; A_0) \sim \log M(r; A_0)$ for $r \notin E_2$ where $m(E_2) < \infty$, this implies $m(I_{A_0}(r)) \to \infty$. Combining (2), (6) with (8), yields

$$M(r, A_0)^c \leq \left|A_0\left(re^{i\theta}\right)\right| \leq \left|\frac{f^{(n)}(z_n)}{f(z_n)}\right| + \sum_{j=1}^{n-1}\left|\frac{f^{(j)}(z_n)}{f(z_n)}\right| A_j(z_n) + \cdots + \left|\frac{f'(z_n)}{f(z_n)}\right| \leq (n-1)(1+o(1))r_n^{2\rho(f)}$$

for $r \notin E_1 \cup E_2 \cup [0,10\ r \to \infty$ and $\theta \in I_{A_s} \setminus I_{A_0}$. Since $A_0(z)$ is transcendental, we obtain a contradiction.

Theorem 6 Let $A_j(z)$ be EFs that are CRG and $E = \{\theta \in [0,2\pi): h(\theta) = 0\}$ is with $m(E) = 0$, and let $A_0(z)$ be EF extremal for DC. Then any nonzero solution of (1) is with infinite order.

Proof Assume that $f \neq 0$ is a solution of (1) has $\rho(f) < \infty$. We have 2 cases.

Case 1. Suppose $m(E^*) = 0$, then the indicator of $A_s(z)$ is with $h(\theta) > 0$ for every $\theta \in [0, 2\pi) \setminus E^*$. The arguments are similar as in Case 1 in the previous theorem.

Case 2. Assume $m(E^*) > 0$, then there exist a sectors satisfy $A_s(z)$'s indicator, $1 \leq s \leq n-1$ is with $h(\theta) < 0$. Hence, there is $I_{A_s} \subseteq [0, 2\pi)$ where I_{A_s} is an interval s. t. $h(\theta) < 0, \theta \in I_{A_s}$. We choose a ray $\arg z = \theta^*$ belongs to I_{A_s}. Using Lemma 1, there is $E_1 \subseteq (1, \infty)$ has $m(E_1) < \infty$ s. t. (12) holds for all z with $|z| = r \notin E_1 \cup [\,[0, 1]$. We have the following two subcases:

Subcase. 1. Assume that $\arg z = \theta^*$ is a Borel direction with order $\rho(A_0)$ of $A_0(z)$. Given $\eta > 0$ small enough so that $(\theta^* - \eta, \theta^* + \eta) \subseteq I_{A_s}$ and $2\eta < \frac{\pi}{\rho(A_s)}$. Choose $(\theta^* - \eta < \varphi_1 < \theta^*)$ and $(\theta^* < \varphi_2 < \theta^* + \eta)$, then $\varphi_2 - \varphi_1 < \frac{\pi}{\rho(A_s)}$. By Lemma 6, $L_1: \arg z = \varphi_1$ or $L_2: \arg z = \varphi_2$, say L_2, satisfies (5). Combining (2) with (6) yields

$$\left|A_0\left(r_n e^{i\theta}\right)\right| \leq \left|\frac{f^{(n)}(r_n e^{i\varphi_2})}{f(r_n e^{i\varphi_2})}\right| + \sum_{j=1}^{n-1} A_j(z_n) + \left|\frac{f(r_n e^{i\varphi_2})}{f(r_n e^{i\varphi_2})}\right| \leq (n-1)\left(1+o(1)\right)r_n^{2\rho(f)}, r_n \notin E_1 \tag{9}$$

By Lemma 3 and (5), there is $G \subseteq \mathbb{R}$ has $m(G) = \infty$ s. t. $\left|A_0\left(re^{i\theta}\right)\right| > \exp\{r^{\rho(A_0)-\epsilon}\}, r \in G, \theta \in [0, 2\pi)$ and $\epsilon > 0$ small enough. Combine this and (9), we get

$$\exp\{r^{\rho(A_0)-\epsilon}\} < \exp\{r^{\rho(A_0)-\epsilon}\} \leq (n-1)\left(1+o(1)\right)r_n^{2\rho(f)}, \ r \in G \setminus E_1$$

This is a contradiction.

Subcase. 2. Assume $\arg z = \theta^*$is not a Borel direction with $\rho(A_0)$ of $A_0(z)$. We take $\delta > 0$ enough small s. t. $(\theta^* - \delta, \theta^* + \delta) \subseteq I_{A_s}$. Using Lemma 4, we get

$$\lim_{|z|\to\infty} \frac{\log\log|A_0(z)|}{\log r} = \rho(A_0), z \in S(\theta^* - \delta, \theta^* + \delta)\backslash E$$

and z holds $\lim_{r\to\infty} m(S(\theta^* - \delta, \theta^* + \delta); r, \infty) \cap E) = 0$. A similar argument used in Subcase 1 we get

$$\exp\{r^{\rho(A_0)-\epsilon}\} < |A_0(re^{i\theta})| \le (n-1)(1+o(1))r_n^{2\rho(f)}, z = re^{i\theta} \in S(\theta^* - \delta, \theta^* + \delta)\backslash E, r \in G\backslash E_1$$

This is a contradiction. Thus, the proof is completed.

CONCLUSIONS

In this research, we have dealt with finding some necessary conditions that must be placed on the coefficients of homogeneous linear CDEs in order to prove that for each nonzero solution to such equations has an infinite order. We have used Nevanlina's theory of complex differential equations to study this type of equations and it proved its strength and effectiveness in area.

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