Numerical Approach for Evaluation of Surface Integrals in Polygonal Domain by Gauss Legendre Quadrature and Generalized Gaussian Quadrature Method

A.M. Yogitha[1], K.T. Shivaram[2*], S. Kiran[3]

[1,2]Department of Mathematics, Dayananda Sagar College of Engineering, Bangalore-560078, India; E-mail: shivaramktshiv@gmail.com

[3]Department of Mathematics, Nitte Meenakshi Institute of Technology, Bangalore-560064, India.

Abstracts: We present a new approach for the numerical integration of arbitrary functions over polygonal region, by applying two kinds of quadrature method Gauss Legendre quadrature and Generalized Gaussian quadrature method, the polygonal region is divided into arbitrary triangles, the sides of each triangle is noted as equation of straight line by joining two end vertices, this approach is used to further reduces the integral equations, numerical integration of rational and irrational functions are approximated computationally, we illustrate several numerical examples to shows the accuracy of the present method

Keywords: Numerical Integration, Polygonal Domain, Rational Function, Irrational Function, Quadrature Methods

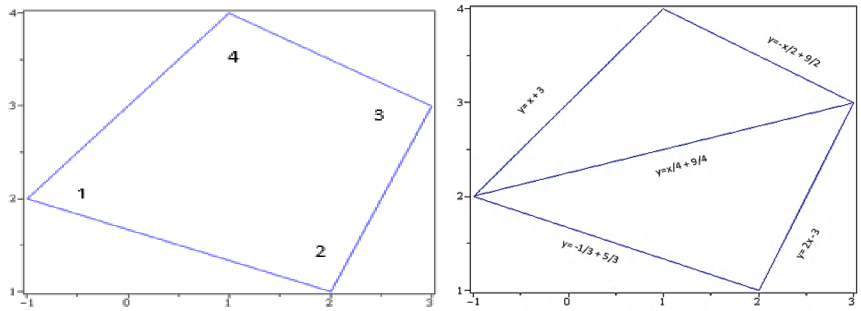
1. INTRODUCTION

Finite element method is a numerical procedure to solve many engineering problems arrive in all branches of engineering and applied sciences, in finite element analysis design the object are made by the finite element equations, some errors is occurs in finite element analysis to reduce these errors by using numerical integration based on quadrature method, numerical integration of rational and irrational functions over polygonal domain is required in many computational methods such as line integral method [1-2], spline element method [3-4], cubature method [5], Finite element mesh generation method [7-8], polygonal domain is discretized into 4, 8 and 12 noded quadrilateral elements [6],[9-11]. In this paper we demonstrated that the polygonal is divided into triangles and each sides of the triangle is noted by equations of the straight line joining the two end vertices, using the transformation formula the limits of the integrational equations are transformed to square region, the method is developed in this paper can be used as Gauss-Legendre quadrature and Generalized Gaussian quadrature rule, we provide an CAS program in Mathematica to implement the above method, several numerical examples are provided to test the accuracy and efficiency of the proposed method,

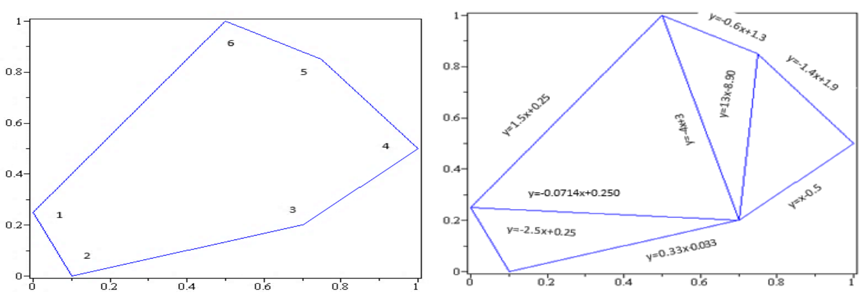
The paper is organized as follows. In Section 2. Selecting the polygonal region and it is discretized into triangles and sides of every triangle is represented as straight line equations, in Section 3. Efficient quadrature formula is developed for solving two dimensional integral problems, in Section 4. Numerical examples are provided and compare the results with other existing methods**.**

2. POLYGONAL REGION

The arbitrary function $f(s,t)$ are integrated over a polygonal domain are shown in Fig.1a and 1b



(a)



(b)

Figure 1. Test on polygonal domain is partition into arbitrary triangles (a) and (b)

Our approach is able to handle the numerical integration of arbitrary functions over the polygonal domain and fix the end points of the boundrie vertices are listed in table. 1

Table.1 Boundrie vertices of polygonal region

Polygonal domain in Fig.1	Vertices	S	t
(a)	1	-1	2
	2	2	1
	3	3	3
	4	1	4
(b)	1	0	0.25
	2	0.1	0
	3	0.7	0.2
	4	1	0.5
	5	0.75	0.85
	6	0.5	1

3. INTEGRATION OF ARBITRARY TRIANGLES OVER A POLYGONAL REGION

The polygonal domain is divided into triangles and each edges of the triangle is associated with straight line equation, the sum of the triangle area contribute to the integral of the function f over the triangle region, the polygonal domain in Fig 1 are used to verify the accuracy of proposed method to calculate the integrals over polygonal region, the integral of arbitrary function f over a polygonal region is defined as

$$I = \iint_D f(s,t)\, ds\, dt = \iint_D f(s,t) ds\, dt$$

Where D is the polygonal domain are taken in Fig.1

$$I = \iint_D \; f(s,t)\, ds\, dt = \sum_{\Delta} \iint \quad f(s,t)\, ds\, dt$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} f\big(s_i, t_j\big)\, |J| \;\, w_i\, w_j$$

s_i, t_j are Gaussian points, J is the jacobian and w_i, w_j are its weights

3.1 Polygonal region 1.

The integral of f over the polygonal region Fig. 1a can be generalized to

$$I_1 = \iint_D \; f(s,t)\, ds\, dt = \int_{-1}^{1} \int_{\frac{s}{4}+\frac{9}{4}}^{s+3} f(s,t) dsdt + \int_{1}^{3} \int_{\frac{s}{4}+\frac{9}{4}}^{-\frac{s}{2}+\frac{9}{2}} f(s,t) dsdt$$
$$+ \int_{-1}^{2} \int_{-\frac{s}{3}+\frac{5}{3}}^{\frac{s}{4}+\frac{9}{4}} f(s,t) dsdt + \int_{2}^{3} \int_{2s-3}^{\frac{s}{4}+\frac{9}{4}} f(s,t) dsdt$$

Using transformation formula xy- region is transform to 2-square region by choosing
$s = (b-a)s_i + a$ and $t = (d-c)t_j + c$ and $dsdt = |J|\, ds_i\, dt_j$

$$I_1 = \int_0^1 \int_0^1 f(s_1, t_1)\; J_1 ds_1 dt_1 + \int_0^1 \int_0^1 f(s_2, t_2)\, J_2 ds_2 dt_2$$
$$+ \int_0^1 \int_0^1 f(s_3, t_3)\, J_3 ds_3 dt_3 + \int_0^1 \int_0^1 f(s_4, t_4)\, J_4 ds_4 dt_4$$

$$= \Delta_1 \sum_{i=1}^{n} \sum_{j=1}^{n} f(s_i, t_j)\; |J|\, w_i\, w_j$$

3.2 Polygonal Region 2.

The integral of f over the polygonal region Fig. 1b can be generalized to

$$I_2 = \int_0^{0.1} \int_{-2.5s+0.5}^{-0.0714s+0.250} f(s,t) dsdt + \int_{0.1}^{0.7} \int_{0.333s-0.033}^{-0.0714s+0.250} f(s,t) dsdt$$
$$+ \int_{0.7}^{0.75} \int_{s-0.5}^{13s-8.90} f(s,t) dsdt + \int_{0.75}^{1} \int_{s-0.5}^{-1.4s+1.9} f(s,t) dsdt$$

$$+ \int_{0.5}^{0.7} \int_{-4s+3}^{-0.6s+1.3} f(s,t) dsdt \;\; + \int_{0.7}^{0.75} \int_{13s-8.90}^{-0.6s+1.3} f(s,t) dsdt$$

$$+ \int_{0}^{0.5} \int_{-0.0714s+0.250}^{1.5s+0.25} f(s,t) dsdt \;\; + \int_{0.5}^{0.7} \int_{-0.0714s+0.250}^{-4s+3} f(s,t) dsdt$$

$$I_2 = \int_0^1 \int_0^1 f(s_1, t_1)\; J_1 ds_1 dt_1 + \int_0^1 \int_0^1 f(s_2, t_2)\, J_2 ds_2 dt_2$$

$$+ \int_0^1 \int_0^1 f(s_3, t_3)\, J_3 ds_3 dt_3 + \int_0^1 \int_0^1 f(s_4, t_4)\, J_4 ds_4 dt_4$$

$$+ \int_0^1 \int_0^1 f(s_5, t_5)\, J_5 ds_5 dt_5 + \int_0^1 \int_0^1 f(s_6, t_6)\, J_6 ds_6 dt_6$$

$$+ \int_0^1 \int_0^1 f(s_7, t_7)\, J_7 ds_7 dt_7 + \int_0^1 \int_0^1 f(s_8, t_8)\, J_8 ds_8 dt_8$$

$$= \Delta_2 \sum_{i=1}^{n} \sum_{j=1}^{n} f(s_i, t_j) \ |J| \ w_i \, w_j$$

4. NUMERICAL RESULTS

The implementation expressed in section 3 is applied to test the problems to determine its versatility and capabilities to accurately and effectively integrate the arbitrary functions in polygonal region by use of Generalized Gaussian quadrature and Gauss Legendre quadrature rule of various order N, the integral domain of Fig. 1b, we test the following functions are considered in [4] as

$$f_1(s,t) = e^{-100((s-0.5)^2+(t-0.5)^2)},$$

$$f_2(s,t) = \sqrt{(s-0.5)^2 + (t-0.5)^2},$$

$$f_3(s,t) = |s^2 + t^2 - \frac{1}{4}|,$$

$$f_4(s,t) = \sqrt{|3 - 4s - 3t|}$$

In Fig. 1a, we consider the following functions are given in [5] as

$$f_5(s,t) = 1/\sqrt{(s+t)},$$

The integrals of the functions f_1 to f_5 over the polygonal domain of Fig 1a and 1b. could be computed by Mathematica program, the numerical results are presented in Table 1. up to order 20.

Integral values given in [4] & [5]	Order N	Generalized quadrature Method	Gauss- Legendre Quadrature Method
f_1 =0.03141452863239306088	N=5	0.0265169799046925	0.02972155727338481
	N=10	0.0307615543202533	0.03141701422536620
	N=15	0.0314105516979906	0.03141615543202533
	N=20	0.0314145286314568	0.03141452863236155
f_2 =0.15682512558627589171	N=5	0.1565972246999544	0.15665962257871272
	N=10	0.1568232217662015	0.15682528865731230
	N=15	0.1568251255373211	0.15682512558609931
	N=20	0.1568251255862359	0.15682512558627574
f_3 =0.19906254943518905316	N=5	0.1972276075565620	0.19825297632888217
	N=10	0.1990580321567811	0.19906233564300935
	N=15	0.1990625445366901	0.19906254947056430
	N=20	0.1990625494351805	0.19906254943518326
f_4 =0.54536805005417548157	N=5	0.5287136832797267	0.5366094740985211
	N=10	0.5453674432161164	0.5452670094332189
	N=15	0.5453868077412785	0.5453868050921335
	N=20	0.5453868050056610	0.5453868050054399
f_5 =3.549613026789710	N=5	3.549606530240252	3.549622707529867
	N=10	3.549613053208934	3.549613375332110
	N=15	3.549613026893202	3.549613026785893
	N=20	3.549613026789710	3.549613026789713

CONCLUSIONS

We developed the quadrature based two kinds of numerical methods Gauss Legendre quadrature and Generalized gaussian quadrature for the solution of various kinds of integral equations in polygonal region, the present approach described as the polygonal domain is discretized into triangles, the sum of the triangle area contribute to the integral of the functions, which is an advantage of our approach to compare the numerical results with existing method, this quadrature method has been tested to five integral problems, the results shows our methods is applicable and posses with high accuracy.

REFERENCES

[1] G. Dasgupta, “Integration within polygonal finite elements’’, Journal of Aerospace Engineering, 16(1), pp. 9–18, 2003.

[2] P.L.Powar,T. Rishabh, Vishnu Narayan Mishra, “Extension of Dasgupta’s technique for Higher degree approximation’’, Univ.Sci. 26(2), pp. 139-157, 2021.

[3] C.J. Li, R.H. Wang, “A new 8-node quadrilateral spline finite element’’, Journal of Computational and Applied Mathematics, 195 pp.54–65, 2006.

[4] Li, C.J, Paola Lamberti and Catterina Dagnino, Numerical integration over polygons using an eight-node quadrilateral spline finite element, Journal of Computational and Applied Mathematics, 233, pp.279-292, 2009.

[5] B. Chin, Eric, Lasserre, B.Jean and N. Sukumar, “Numerical Integration of homogeneous functions on convex and non-convex polygons and polyhedral’’, Journal of Comput. Mech., 56, pp.967-981, 2015.

[6] M.S. Islam, M. Alamgir Hossain, “Numerical integration over an arbitrary quadrilateral region’’, Applied Mathematics and Computation, 210, pp.515-524, 2009

[7] H.R. Jyothi, K.T. Shivaram, “ A new finite element mesh generation technique for solving surface integrals over polygonal domain’’, AIP Conference Proceedings, 2235(1), pp. 020001-020006, 2020.

[8] K.T. Shivaram, H.R. Jyothi, “Mesh generation for numerical integration of arbitrary function over polygonal domain by finite element method’’, IOP Conference series Material Science and Engineering, 577(1), pp. 1-8, 2019.

[9] K.T. Shivaram, A. Yogitha, “Numerical Integration of Arbitrary Functions over a Convex and non-convex polygonal region by quadrature method’’, J. Mat. Comp. Sci., 6, pp.1177- 1186, 2016.

[10] A.M. Yogitha, K.T. Shivaram, “Numerical Integration of arbitrary Function over a Convex and Non convex polygonal domain by Eight noded Linear quadrilateral Finite Element Method’’, Aus.J. Bas. App. Sci.,10, pp. 104 – 110, 2016.

[11] K.T. Shivaram, H.R. Jyothi, A.M Yogitha, “An Accurate Evaluation of Integrals in Convex and Non convex Polygonal Domain by Twelve Node Quadrilateral Finite Element Method”, Int. J. Comp.App. Mat. 12, pp. 233 – 241, 2017.

[12] K.T. Shivaram, H.R. Jyothi, “Finite element approach for numerical integration over family of eight node linear quadrilateral element for solving Laplace equation’’, Material Today Proceedings, 46(9), 4336-4340. 2021.

DOI: https://doi.org/10.15379/ijmst.v10i1.2791