Solution for 3rd order Cauchy Difference Equation in Free Monoid

M Pradeep
Department of Mathematics
Arignar Anna Government Arts College, Cheyyar-604407
pradeepmprnet12@gmail.com

Abstract— Let $k: C \to D$ be a function, where (C,.) is a Group and (D,+) is an Abelian Group. In this article, the Third Order Cauchy difference of $k{:}E^{(3)} k(c_1,c_2,c_3,c_4) = k(c_1c_2c_3c_4) - k(c_1c_2c_3) - k(c_1c_3c_4) - k(c_1c_2c_3c_4) - k(c_2c_3c_4) + k(c_1c_2) + k(c_1c_3) + k(c_1c_4) + k(c_2c_3) + k(c_2c_4) + k(c_3c_4) - k(c_1) - k(c_2) - k(c_3) - k(c_4)$ $\forall c_1,c_2,c_3,c_4 \in C$ is studied. We give solutions of $E^{(3)}k=0$ on Free Monoid.
Keywords—Abelian Group, Cauchy Difference Equation, Free Monoid, Free Group, Group,

1 INTRODUCTION

We know that from [1] Jenson's equation

$$k(c+d)+k(c-d)=2k(c) \qquad (1.1)$$

with $k(0)=0$, is equal to Cauchy's equation

$$k(c+d)=k(c) + k(d)$$

in the real line. Let (C,.) is a Group, (D,+) is an Abelian Group. Let $e \in C$ and $0 \in D$ are identity.
For a function $k: C \to D$, its Cauchy Difference Equation $E^{(m)}k$, is define

$$E^{(0)}h = h, \qquad (1.2)$$

$$E^{(1)} k(c_1c_2) = k(c_1c_2) - k(c_1) - k(c_2) \qquad (1.3)$$

$$E^{(m+1)} k(c_1,c_2,\ldots,c_{m+2}) = E^{(m)}k(c_1,c_2,c_3,\ldots,c_{m+2}) - E^{(m)} k(c_1,c_3,\ldots,c_{m+2}) - E^{(m)} k(c_2,c_3,\ldots,c_{m+2}) \qquad (1.4)$$

$E^{(1)}k$ denoted as Ek. In [6], by using the reduction formulas and relations, in [4,5], the general solution of Cauchy Difference Equation given on free groups.

We consider the following functional equation, in this article

$$k(c_1c_2c_3c_4) - k(c_1c_2c_3) - k(c_1c_3c_4) - k(c_1c_2c_4) - k(c_2c_3c_4) + k(c_1c_2) + k(c_1c_3) + k(c_1c_4) + k(c_2c_3) + k(c_2c_4) + k(c_3c_4)$$
$$-k(c_1) - k(c_2) - k(c_3) - k(c_4)=0 \quad \forall c_1,c_2,c_3,c_4 \in C \qquad (1.5)$$

From (1.4) that (1.5) is

$$E^{(3)}k = 0$$

The aim of this article is to find the solutions of (1.5) on Free Monoid.
The solution of (1.5) define

$$\text{Ker } E^{(3)}(C,D) = \{k : C \to D \mid k \text{ satisfies } (1.5)\} \qquad (1.6)$$

Remark 1

1. Ker $E^{(3)}$ (C,D) is an Abelian Group under the pointwise addition of functions;
2. Hom(C,D) ≤ Ker $E^{(3)}$ (C,D)

2. Properties of Solutions

Lemma 1 If $k \in$ Ker $E^{(3)}$ (C,D) then

$$k(e) = 0, \qquad (2.1)$$

$$Ek(c_1,c_2) = 0, \text{ when } c_1 = e \text{ or } c_2 = e \qquad (2.2)$$

$$E^2k(c_1,c_2,c_3) = 0 \text{ when } c_1 = e \text{ or } c_2 = e \text{ or } c_3 = e \qquad (2.3)$$

$$E^2k \text{ is a Homomorphism w.r.t every variable} \qquad (2.4)$$

$$k(c^m) = mk(c) + \binom{m}{2} Ek(c,c) + \binom{m}{3} E^2k(c,c,c) \qquad (2.5)$$

for all $c,c_1,c_2,c_3 \in C$ and $m \in Z$.

Proof:

In (1.5) Put $c_1 = e \implies$ (2.1).
We can Prove (2.2)-(2.3) from (2.1)
Also,from the definition of E^2k,

$$E^2k(c_1,c_2c_3,c_4) = k(c_1c_2c_3c_4) - k(c_1c_2c_3) - k(c_1c_4) - k(c_2c_3c_4) + k(c_1) + k(c_2c_3) + k(c_4)$$

and

$$E^2k(c_1,c_2,c_4) + E^2k(c_1,c_3,c_4) = k(c_1c_2c_4) - k(c_1c_2) - k(c_1c_4) - k(c_2c_4) + k(c_1) + k(c_2) + k(c_4) + k(c_1c_3c_4) - k(c_1c_3) - k(c_1c_4) - k(c_3c_4) + k(c_1) + k(c_3) + k(c_4)$$

$\Rightarrow E^2k(c_1,c_2,c_3,c_4) - E^2k(c_1,c_2,c_4) - E^2k(c_1,c_3,c_4) = E^3k(c_1,c_2, c_3,c_4) = 0$

$\Rightarrow$ $E^2k(c_1, \cdot ,c_3)$ is a homomorphism.

Similar way we can prove $E^2k (\cdot ,c_2,c_3)$ and $E^2k(c_1,c_2, \cdot)$ are homomorphism.

Hence (2.4) proved.

To prove (2.5). This is Obviously true for m = 0,1,2 from (2.1) and from the definition of *Ek* and E^2k.

Assume (2.5) true for all m≤3 $m \in N$, then

$$k(c^m) = k(c^{m-2} \cdot c \cdot c)$$
$$= 2k(c^{m-1}) + k(c^2) - k(c^{m-2}) - 2k(c) + E^2k(c^{m-2}, c, c)$$
$$= 2[(m-1)\, k(c) + (m-1)C_2Ek(c, c) + (m-1)C_3E^2k(c, c, c)] + 2k(c) + Ek(c, c)$$
$$-[(m-2)k(c) + (m-2)\, C_2Ek(c, c) + (m-2)\, C_3E^2k(c, c, c)] - 2k(c) + (m-2)E^2k(c, c, c)$$
$$= mk(c) + mC_2Ek(c, c) + mC_3E^2k(c, c, c)$$

$\Rightarrow$ (2.5) proved $\forall m \geq 0$.

For m>0,

$$Ek(c^m, c^{-m}, c^m) = k(c^m) - k(e) - k(c^{2m}) - k(e) + k (c^{-m}) + k(c^m)$$
$$= 3k(c^m) + k(c^{-m}) - k(c^{2m})$$
$$\Rightarrow k(c^{-m}) = k(c^{2m}) - 3k(c^m) + E^2k(c^m, c^{-m}, c^m)$$
$$= [2mk(c) + (2m)C_2Ek(c, c) + (2m)\, C_3E^2k(c, c, c)] - 3[mk(c)$$
$$+ mC_2Ek(c,c) + m\, C_3E^2k(c, c, c)] - (m^3)E^2k(c, c, c)$$
$$= -mk(c) + (-m)C_2Ek(c,c) + (-m)\, C_3E^2k(c, c, c)$$

Hence (2.5) proved for m<0.

Remark 2 The following are pairwise equivalent for k:C→D

(i) $k \in$ Ker $E^{(3)}$ (C, D);

(ii) $E^2k (\cdot ,c_2,c_3)$ is a Homomorphism;

(iii) $E^2k(c_1, \cdot ,c_3)$ is a Homomorphism;

(iv) $E^2k(c_1,c_2, \cdot)$ is a Homomorphism

$\Rightarrow$ E^2k is a Homomorphism w.r.t every variable.

Since D is abelian, (2.4) implies that E^2k can be factored through the abelianized C^{abc}

Remark 3: Let k: C→ D be a function. For any fixed $c_4 \in$ C, consider the function h(c_1) := Ek(c_1, c_4).Taking the Cauchy difference of h twice we get $E^2h(c_1,c_2,c_3) = h(c_1c_2c_3) - h(c_1c_2) - h(c_1c_3) - h(c_2c_3) + h(c_1) + h(c_2) + h(c_3)$. Since h = Ek(. ,$c_4$) we may write that as

$$E^2Ek(.\ ,c_4)(c_1,c_2,c_3) = Ek(.\ ,c_4)(c_1c_2c_3) - Ek(.\ ,c_4)(c_1c_2) - Ek(.\ ,c_4)(c_1c_3) - Ek(.\ ,c_4)(c_2c_3) + Ek(.\ ,c_4)(c_1) + Ek(.\ ,c_4)(c_2) + Ek(.\ ,c_4)(c_3)$$
$$= Ek((c_1c_2c_3), c_4) - Ek((c_1c_2), c_4) - Ek((c_1c_3), c_4) - Ek((c_2c_3), c_4) + Ek((c_1), c_4) + Ek((c_2), c_4) + Ek((c_3), c_4)$$
$$= E^3(c_1,c_2,c_3,c_4)$$

Similarly $\Rightarrow$ $E^{(n)}Ek(.\ ,c_{m+2})(c_1,c_2,\ldots\ldots,c_{n+1}) = E^{(n+1)}k(c_1,c_2,\ldots\ldots,c_{n+1},c_{n+2})$ for all higher orders n.

Lemma 2 For any function k:C→D and $t \in \mathbb{N}$, the fowllowing result is valid;

$$k(c_1c_2\ldots c_t) = \sum_{n\leq t}\sum_{1\leq j_1<j_2<\ldots<j_n\leq t}E^{(n-1)}k((c_{j_1},c_{j_2},\ldots\ldots,c_{j_n}) \tag{2.6}$$

Proposition 1 If k ∈ Ker$E^{(3)}$(C,D).Then

$$k(c_1^{m_1}c_2^{m_2}\ldots c_t^{m_t}) = \sum_{1\leq j\leq t}[m_jk(c_j) + m_jC_2Ek(c_j,c_j) + m_jC_3E^2k(c_j,c_j,c_j)] + \sum_{1\leq j_1\leq j_2\leq t}m_{j_1}m_{j_2}Ek(c_{j_1},c_{j_2})$$
$$+\sum_{1\leq j_1\leq j_2\leq j_3\leq t}m_{j_1}m_{j_2}m_{j_3}E^2k(c_{j_1},c_{j_2},c_{j_3}) \tag{2.7}$$

for $m_j \in \mathbb{Z}$ and all $c_j \in C$, j=1,2,....t such that $c_i \neq c_{i+1}$, i= 1,2, ...,t-1

Proof

Replacing c_j in (2.6) by $c_j^{m_j}$, we have

$$k(c_1^{m_1}, c_2^{m_2}\ldots\ldots c_t^{m_t}) = \sum_{n\leq t}\sum_{1\leq j_1<j_2<\cdots<j_n\leq t}E^{(n-1)}k(c_{j1}^{m_{j1}}, c_{j2}^{m_{j2}},\ldots . c_{jn}^{m_{jn}})$$

$E^{(n-1)}$ k=0 for n≥4 gives

$$k(c_1^{m_1}c_2^{m_2}\ldots c_t^{m_t}) = \sum_{1\leq j\leq t}k(c_j^{m_j}) + \sum_{1\leq j_1<j_2\leq t}Ek(c_{j_1}^{m_{j_1}},c_{j_2}^{m_{j_2}}) + \sum_{1\leq j_1<j_2<j_3\leq t}E^2k(c_{j_1}^{m_{j_1}},c_{j_2}^{m_{j_2}},c_{j_3}^{m_{j_3}})$$

$$\Rightarrow k(c_1^{m_1}c_2^{m_2}\ldots c_t^{m_t}) = \sum_{1\leq j\leq t}[m_jk(c_j) + m_jC_2Ek(c_j,c_j) + m_jC_3E^2k(c_j,c_j,c_j)] + \sum_{1\leq j_1<j_2\leq t}m_{j_1}m_{j_2}Ek(c_{j_1},c_{j_2})$$
$$+\sum_{1\leq j_1<j_2<j_3\leq t}m_{j_1}m_{j_2}m_{j_3}E^2k(c_{j_1},c_{j_2},c_{j_3})$$

Therefore (2.7) is proved

3 Solution in a free Monoid

Since every Free Monoid can be embedded in a free group.

First solve (1.5) for the Free Monoid C on a one letter c.

Theorem 1 Let C is the Free Monoid on single character c. Then k∈Ker$E^{(3)}$(C,D)

Iff k(c^m)= mk(c)+mC_2Ek(c,c)+mC_3E^2k(c,c,c) for all m ∈ W (3.1)

Proof

⇒ From (2.5) in lemma 1,

k(c^m)=mk(c)+mC_2Ek(c,c)+mC_3E^2k(c,c,c)

⇐ Take k(c^m)= mk(c)+mC_2Ek(c,c)+mC_3E^2k(c,c,c) for all m ∈ W on C =< c >.

To Prove k ∈ Ker$E^{(3)}$(C,D).

i.e., we want to prove E^2k is a homomorphism w.r.t every variable

Assume

$$u=c^q, v=c^r, w=c^s$$

be any 3 elements of C.

E^2k(u,v,w)=E^2k(c^q, c^r, c^s)

=k(c^{q+r+s})−k(c^{q+r})−k(c^{q+s}) −k(c^{r+s})+k(c^q)+k(c^r)+k(c^s)

=[(q+r+s)k(c)+(q+r+s)C_2Ek(c,c)+(q+r+s)C_3E^2k(c,c,c)]

−[(q+r)k(c)+(q+r)C_2Ek(c,c)+(q+r)C_3E^2k(c,c,c)]

−[(q+s)k(c)+(q+s)C_2Ek(c,c)+(q+s)C_3E^2k(c,c,c)]

−[(r+s)k(c)+(r+s)C_2Ek(c,c)+(r+s)C_3E^2k(c,c,c)]

+[qk(c)+qC_2Ek(c,c)+qC_3E^2k(c,c,c)]

+[rk(c)+rC_2Ek(c,c)+rC_3E^2k(c,c,c)]

+[sk(c)+sC_2Ek(c,c)+sC_3E^2k(c,c,c)

From lengthly simplification,

E^2k(c^q, c^r, c^s)=qrsE^2k(c,c,c)

⇒ E^2k is a homomorphism w.r.t every variable

Finally, for the Free Monoid on an alphabet $\langle\mathcal{C}\rangle$ with$|\mathcal{C}| \geq 2$, we discuss Some special solution of (1.5).

A letter c ∈ $\mathcal{C}$

c = $c_1^{m1}c_2^{m2} \ldots c_t^{mt}$, *where* $c_j \in \mathcal{C}, m_j \in \mathbb{W}$

For each fixed $c \in C$ and fixed pair of distinct $p,q \in C$, define the functions T_1 , T_2 , T_3 , T_4 , T_5 :

$$T_1(c;p) = \sum_{c_j=p} m_j \qquad (3.2)$$

$$T_2(c;p,q) = \sum_{j<i, c_j=p, c_i=q} m_j m_i \qquad (3.4)$$

$$T_3(c;p,q) = \sum_{j>i, c_j=p, c_i=q} m_j m_i \qquad (3.5)$$

$$T_4(c;p,q,r) = \sum_{h<j<i, c_h=p, c_j=q, c_i=r} m_h m_j m_i \qquad (3.6)$$

$$T_5(c;p,q,r) = \sum_{h>j>i, c_h=p, c_j=q, c_i=r} m_h m_j m_i \qquad (3.7)$$

with reference from [4,5], T_1, T_2, T_3, T_4, T_5 are well defined.

Also, T_1, T_2, T_3, T_4, T_5 satisfy:

$$T_1 \quad is \quad additive: T_1(cd;p) = T_1(c;p) + T_1(d;p) \qquad (3.8)$$

$$T_1(c,p)T_1(c,q) = T_2(c;p,q) + T_3(c;p,q) \qquad (3.9)$$

$$T_1(c,p)T_1(c,q)T_1(c,r) = T_4(c;p,q,r) + T_5(c;p,q,r) \qquad (3.10)$$

$$T_3(c;p,q) = T_2(c;q,p) \qquad (3.11)$$

$$T_4(c;p,q,r) = T_5(c;r,q,p) \qquad (3.12)$$

Proposition 2. *For any fixed pair of distinct letters p,q in* C ,

(i) $T_1(\ ;p) \in KerE^{(3)}(C,W)$;

(ii) $T_2(\ ;p,q) \in KerE^{(3)}(C,W)$;

(iii) $T_3(\ ;p,q) \in KerE^{(3)}(C,W)$;

(iv) $T_4(\ ;p,q,r) \in KerE^{(3)}(C,W)$;

(v) $T_5(\ ;p,q,r) \in KerE^{(3)}(C,W)$;

Proof.

From (3.8), (i) proved

For (ii). Let write $c_1, c_2, c_3, c_4 \in$ C is of the form

$$c_1 = c_{11}^{s_{11}} c_{12}^{s_{12}} \ldots c_{1t}^{s_{1t}},$$

$$c_2 = c_{21}^{s_{21}} c_{22}^{s_{22}} \ldots c_{2t}^{s_{2t}},$$

$$c_3 = c_{31}^{s_{31}} c_{32}^{s_{32}} \ldots c_{3t}^{s_{3t}},$$

$$c_4 = c_{41}^{s_{41}} c_{42}^{s_{42}} \ldots c_{4t}^{s_{4t}},$$

Then

$$T_2(c_1c_2c_3c_4;p,q) = \sum_{1j<1i, c1j=p, c1i=q} s_{1j}s_{1i} + \sum_{2j<2i, c2j=p, c2i=q} s_{2j}s_{2i} + \sum_{3j<3i, c3j=p, c3i=q} s_{3j}s_{3i} + \sum_{4j<4i, c4j=p, c4i=q} s_{4j}s_{4i} + \sum_{c1j=p, c2i=q} s_{1j}s_{2i} + \sum_{c1j=p, c3i=q} s_{1j}s_{3i} + \sum_{c1j=p, c4i=q} s_{1j}s_{4i} + \sum_{c2j=p, c3i=q} s_{2j}s_{3i} + \sum_{c2j=p, c4i=q} s_{2j}s_{4i} + \sum_{c3j=p, c4i=q} s_{3j}s_{4i}$$

$$T_2(c_1c_2c_3;p,q) = \sum_{1j<1i, c1j=p, c1i=q} s_{1j}s_{1i} + \sum_{2j<2i, c2j=p, c2i=q} s_{2j}s_{2i} + \sum_{3j<3i, c3j=p, c3i=q} s_{3j}s_{3i} + \sum_{c1j=p, c2i=q} s_{1j}s_{2i} + \sum_{c1j=p, c3i=q} s_{1j}s_{3i} + \sum_{c2j=p, c3i=q} s_{2j}s_{3i}$$

$$T_2(c_1c_2c_4;p,q) = \sum_{1j<1i, c1j=p, c1i=q} s_{1j}s_{1i} + \sum_{2j<2i, c2j=p, c2i=q} s_{2j}s_{2i} + \sum_{4j<4i, c4j=p, c4i=q} s_{4j}s_{4i}$$

$$+ \sum_{c1j=p,c2i=q} S_{1j}S_{2i} + \sum_{c1j=p,c4i=q} S_{1j}S_{4i} + \sum_{c2j=p,c4i=q} S_{2j}S_{4i}$$

$$T_2(c_1c_3c_4;p,q) = \sum_{1j<1i,c1j=p,c1i=q} S_{1j}S_{1i} + \sum_{3j<3i,c3j=p,c3i=q} S_{3j}S_{3i} + \sum_{4j<4i,c4j=p,c4i=q} S_{4j}S_{4i}$$

$$+ \sum_{c1j=p,c3i=q} S_{1j}S_{3i} + \sum_{c1j=p,c4i=q} S_{1j}S_{4i} + \sum_{c3j=p,c4i=q} S_{3j}S_{4i}$$

$$T_2(c_2c_3c_4;p,q) = \sum_{2j<2i,c2j=p,c2i=q} S_{2j}S_{2i} + \sum_{3j<3i,c3j=p,c3i=q} S_{3j}S_{3i} + \sum_{4j<4i,c4j=p,c4i=q} S_{4j}S_{4i}$$

$$+ \sum_{c2j=p,c3i=q} S_{2j}S_{3i} + \sum_{c2j=p,c4i=q} S_{2j}S_{4i} + \sum_{c3j=p,c4i=q} S_{3j}S_{4i}$$

$$T_2(c_1c_2;p,q) = \sum_{1j<1i,c1j=p,c1i=q} S_{1j}S_{1i} + \sum_{2j<2i,c2j=p,c2i=q} S_{2j}S_{2i} + \sum_{c1j=p,c2i=q} S_{1j}S_{2i}$$

$$T_2(c_1c_3;p,q) = \sum_{1j<1i,c1j=p,c1i=q} S_{1j}S_{1i} + \sum_{3j<3i,c3j=p,c3i=q} S_{3j}S_{3i} + \sum_{c1j=p,c3i=q} S_{1j}S_{3i}$$

$$T_2(c_1c_4;p,q) = \sum_{1j<1i,c1j=p,c1i=q} S_{1j}S_{1i} + \sum_{4j<4i,c4j=p,c4i=q} S_{4j}S_{4i} + \sum_{c1j=p,c4i=q} S_{1j}S_{4i}$$

$$T_2(c_2c_3;p,q) = \sum_{2j<2i,c2j=p,c2i=q} S_{2j}S_{2i} + \sum_{3j<3i,c3j=p,c3i=q} S_{3j}S_{3i} + \sum_{c2j=p,c3i=q} S_{2j}S_{3i}$$

$$T_2(c_2c_4;p,q) = \sum_{2j<2i,c2j=p,c2i=q} S_{2j}S_{2i} + \sum_{4j<4i,c4j=p,c4i=q} S_{4j}S_{4i} + \sum_{c2j=p,c4i=q} S_{2j}S_{4i}$$

$T_2(c_3c_4;p,q) = \sum_{3j<3i,c_{3j}=p,c_{3i}=q} s_{3j}s_{3i} + \sum_{4j<4i,c_{4j}=p,c_{4i}=q} s_{4j}s_{4i} + \sum_{c_{3j}=p,c_{3i}=q} s_{3j}s_{4i}$

$T_2(c_1;p,q) = \sum_{1j<1ic_{1j}=p,c_{1i}=q} s_{1j}\, s_{1i}$

$T_2(c_2;p,q) = \sum_{2j<2i,c_{2j}=p,c_{2i}=q} s_{2j}s_{2i}$

$T_2(c_3;p,q) = \sum_{3j<3i,c_{3i}=p,c_{3j}=q} s_{3j}s_{3i}$

$T_2(c_4;p,q) = \sum_{4j<4i,c_{4j}=p,c_{4i}=q} s_{4j}s_{4i}$

$T_2(c_1c_2c_3c_4;p,q) - T_2(c_1c_2c_3;p,q) - T_2(c_1c_2c_4;p,q) - T_2(c_1c_3c_4;p,q) - T_2(c_2c_3c_4;p,q) - T_2(c_1c_2;p,q) - T_2(c_1c_3;p,q) - T_2(c_1c_4;p,q) - T_2(c_2c_3;p,q) - T_2(c_2c_4;p,q) - T_2(c_3c_4;p,q) - T_2(c_1;p,q) - T_2(c_2;p,q) - T_2(c_3;p,q) - T_2(c_4;p,q) = 0$

$T_2(\cdot;p,q)$ satisfies equations (1.5).

$T_2(\cdot;p,q) \in \text{Ker}E^{(3)}(C,W)$

Similarly we can prove (iii)-(v)

Finally to find the solution of (1.5) on ⟨C⟩, we present C with a liner order <. Every c∈ C is of the form

$$c = y_1^{m_{11}} y_2^{m_{12}} \ldots y_t^{m_{1t}} y_1^{m_{21}} y_2^{m_{22}} \ldots y_t^{2t} \ldots y_1^{m_{s1}} y_2^{m_{s2}} \ldots y_t^{m_{st}} \quad (3.13)$$

Here $y_1<y_2<\ldots\ldots y_t$,

Theorem 2 Assume |C |>2. Suppose that k∈KerE^3(⟨C ⟩,D), then

$$k(c) = \sum_p T_1(c;p)k(p) + \sum_p (T_1(c;p)C_2Ek(p,p) + \sum_p (T_1(c;p)C_3E^2k(p,p,p) + \sum_{p<q}(T_2(c;p,q)Ek(p,q) + \sum_{p<q}(T_3(c;p,q)Ek(q,p) + \sum_{p<q<r}(T_4(c;p,q,r)E^2k(p,q,r) + \sum_{p<q<r}(T_5(c;p,q,r)E^2k(r,q,p)$$

Proof If k satisfies (1.5). For c, with s>1 and t>1, let briefly

$b_j := y_2^{mj2} \ldots y_t^{mjt}$ for j=1,2,….,s

$k(c) = k(y_1^{m11} b_1 y_1^{m21} b_2 \ldots y_1^{ms-1} b_s)$

$= k(y_1^{m11} y_1^{m21} y_1^{m31} b_1 b_2 b_3 \ldots y_1^{ms1} b_s) + \sum_{1<v}[Ek(y_v^{m1v}, y_1^{m21}) - Ek(y_1^{m21}, y_v^{m1v})]$

$+ \sum_{1<v}[E^2k\,(y_v\,(y_v^{m1v}, y_1^{m21}, y_1^{m31}) - E^2k\,(y_1^{31}, y_1^{m21}, y_v^{m1v})]$

$= k(y_1^{T1(c;y1)} b_1 b_2 b_3 \ldots bs) + \sum_{g<j} \sum_{1<v}[Ek(y_q^{mgv}, y_1^{mj1}) - Ek(y_1^{mj1}, y_v^{mgv})]$

$+ \sum_{h<g<j} \sum_{1<v}[E^2\,k(y_q^{mhv}, y_q^{mgv}, y_1^{mj1}) - Ek\,(y_1^{mj1}, y_v^{mgv}, y_q^{mhv})]$

$= k(y_1^{T1(c;y1)} y_2^{T1(c;y2)} \ldots y_t^{T1(c;yt)}) + \sum_{g<j} \sum_{u<v}[Ek(y_q^{mgv}, y_u^{mju}) - Ek(y_u^{mju}, y_v^{mgv})]$

$+\sum_{h<g<j}\sum_{u<v<w}[E^2 k(y_w^{mhw}, y_v^{mgv}, y_u^{mju}) - E^2 k(y_u^{mju}, y_v^{mgv}, y_w^{mhw})]$
$= k(y_1^{T1(c;y1)} y_2^{T1(c;y2)} \ldots y_t^{T1(c;yt)}) + \sum_{g<j}\sum_{u<v} m_{ju}\, m_{gv}[Ek(y_v, y_u) - Ek(y_u, y_v)]$
$+\sum_{h<g<j}\sum_{u<v<w} m_{ju}\, m_{gv}\, m_{hw}[E^2 k(y_w, y_v, y_u) - E^2 k(y_u, y_v, y_w)]$
$= k(y_1^{T1(c;y1)} y_2^{T1(c;y2)} \ldots y_t^{T1(c;yt)}) - \sum_{g<j}\sum_{u<v} m_{ju}\, m_{gv}[Ek(y_u, y_v) - Ek(y_v, y_u)]$
$-\sum_{h<g<j}\sum_{u<v<w} m_{ju}\, m_{gv}\, m_{hw}[E^2 k(y_u, y_v, y_w) - E^2 k(y_w, y_v, y_u)]$

Use (2.7) and (3.9) and (3.10), and Replace y_u, y_v and y_w as p,q and r respectively, we get

$K(c) = \sum_p T_1(c;p)k(p) + \sum_p (T_1(c;p))C_2 Ek(p,p) + \sum_p (T_1(c;p))C_3 E^2 k(p,p,p)$
$+\sum_{p<q} T_2(c;p,q)Ek(p,q) + \sum_{p<q} T_3(c;p,q)Ek(q,p) + \sum_{p<q<r} T_4(c;p,q,r)E^2 k(p,q,r)$
$+\sum_{p<q<r} T_5(c;p,q,r)E^2 k(r,q,p)$

4 Conclusion

The solution for 3rd order Cauchy Difference Equation in Free Monoid has been found. This work can be extend to all types of groups and different type of Cauchy functional Equations.

Acknowledgment

I would like to thank my Supervisor Dr. V Selvan for valuable suggestion and support for this work.

References

[1] Aczel, J: Lectures on Functional Equations and their Applications Academic Press New York (1966).
[2] Baron, K.Kannappan, P: On the Cauchy difference, Aequ. Math. 46 (1993), 112-118.
[3] Brzdek, J: On Cauchy difference on normed space, Abh. Math. Semin. Univ. hamb. 66 (1996) 143- 150.
[4] Ng, CT: Jensen's functional equation on groups, Aequ. Math. 39 (1990), 85-99.
[5] Ng, CT: Jensen's functional equation on groups II, Aequ. Math. 58 (1999), 311-320.
[6] Guo and Li: Solutuions of the third order Cauchy difference equation on Groups, Advances in difference equations 203 (2014).
[7] Brzdek, B:Generalized Cauchy difference functional equations,Aequ. Math. 70 (2005), 154 – 176.
[8] Brzdek, B:Generalized Cauchy difference functional equations,II Proc. Am. Math. Soc. 136 (2008), 3911 -3919.
[9] Ng, CT, Zhao,H: Kernel of the sound order Cauchy difference on groups, Aequ. Math. 86 (2013), 155 – 170.

DOI: https://doi.org/10.15379/ijmst.v10i1.2646