Adjusting control charts for inaccurate measurements

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Abstract: The concept of statistics-based control is nearly a century old. The basic assumption is that the measurement is precise and therefore all calculation based on these data is also precise. Typically, this assumption is not far from reality and thus does not affect the proposed control limits. However, in some cases the rounding-off of measurements may lead to erroneous decisions. This study examines the effects of rounding-off on the average ($\bar{X}$) control lines and suggests a simple method way to design new control lines, based on the conventional average run length (ARL).

Keywords: Quality control, Statistical Process control, Round-off, Measurement, Rounding error

1. INTRODUCTION

The concept of control charts was first introduced by Shewhart [1]. Since this introduction, control charts have been the most popular method for maintaining statistical process control [2, 3]. The most well-known and widely used is the $\bar{X}$-chart [4] for monitoring the mean.

The Shewhart $\bar{X}$-chart is a set of two control limits: the upper control limit (UCL) and the lower control limit (LCL) [5]. These limits are symmetrical around the mean:

$$UCL = \mu + k\frac{\sigma}{\sqrt{n}} \qquad (1)$$

$$LCL = \mu - k\frac{\sigma}{\sqrt{n}}$$

Where μ is the mean, σ is the process' standard deviation, n is the sample size and k is a parameter that is arbitrarily set to balance the rate of false alarms with the requirement to set to the rate of false alarms. Traditionally, a value of 3 is chosen so that the rate of false alarms is 0.0027. A thorough literature review of the development of these control limits and their estimation was prepared by Jensen et al. [6] and need not be repeated here. However, a major problem of all the research and practical usage of the control limits is the underlying (and sometimes not clearly stated) assumption of exact measurement.

The basic assumption is that the gathered data used for the control limits is (nearly) infinitely accurate. This, obviously, is seldom the case as every digital measuring device has its distinct scale step (h) which is the basic level of quantization the equipment has (e.g., a digital thermometer that can measure in steps of 0.1^0C) [7]. Analog devices also suffer from similar problem because the results are rounded to some near scale mark [8]. Furthermore, if a measured variable is real-valued, it is only observed to the nearest kth decimal floating point [9]. The problems of round-off data were not unobserved and it was demonstrated that in some industrial applications where digital instruments are used, data collection often involves relatively crude gaging [10]. The abundance of small, low cost and often inaccurate measuring devices used a part of the Internet of Things (IoT) and the Industry 4.0 revolution creates an even larger problem of rounded-off data [11, 12, 13] and increases the need to deal with such errors. One may conclude that rounding error are simply another type of measurement error. However, these errors differ from the "ordinary" measurement errors, since rounding error depends on the actual value whereas measurement error do not [14, 15].

Obviously, in most cases the effect of rounding-off is limited and negligible. When the ratio of σ to h is high enough, the effects of round-off can be ignored. Gertsbakh [16] observed that a ratio of at least 2 is enough for this policy.

For the best of our knowledge, the first time the need to consider the rounding-off of data was presented was as early as 1898 [17]. In 1957 a rule of thumb to decide when it is reasonable to ignore rounding was suggested [18]. These suggestions were supported by more recent researches [19, 20]. Further research [21] showed that rounded data can harm the results of the control charts that are designed for use with exact measurement. The effects of rounding on R-charts were also examined and demonstrated [22, 23]. The influence of mean quantization on control chart Q [24] was also explored.

Despite the obstacles of rounded-off measurements, researcher have proposed ways to estimate the value of both μ and σ. Originally, Sheppard [17] proposed a method to deal with this problem, but it was later criticized [25] and it was shown that it failed to solve the problem correctly (for various cases). Based on work by Schader and Shmid [26], Gertsbakh [16] developed a maximum-likelihood method to estimate the mean and the standard deviation. In 2012 [27] the method of moments was utilized to evaluate the standard deviation (given a known mean) and proved to be superior to previous methods. Other methods were suggested [28, 29] for the special case of normal distribution (to evaluate both the mean and the variance). The method of moments was improved [31] by combining it with a calibration technique. As for the purposes of this research normality is assumed. The assumption of normal distribution was defended [31] this by pointing out that the random error averages the number of component errors, so that the central limit theorem is applicable and random errors tend to have a normal rather than another distribution.

Although many methods for evaluating the mean and variance no similar attempts have been made to examine the effect of rounding-off on control charts [27]. The purpose of this research is to propose a method for setting control limits for this case.

2. Basic assumptions

Based on the research presented in the introduction section, the following assumptions can be made:

- The mean and standard deviation of the measured process are known and measurable.
- The control process should be designed so as to preserve the (known) mean of the process and provide alerts when deviation from this mean occurs.
- The level of false positive cases (i.e. false alarms) should be maintained at the original conventional level set by Shewhart [1].

Based on these assumptions, a simple mechanism to set control limits is developed in the following sections.

3. Measurement Distribution

Since there is a difference between the real value (X) and the measured value (Y) [27, 30] let us examine the distribution of Y.

As we assumed that X is normally distributed, the probability distribution function of Y is:

$$P(Y=y)=\Phi\left(\frac{yh+\frac{h}{2}-\mu}{\sigma}\right)-\Phi\left(\frac{yh-\frac{h}{2}-\mu}{\sigma}\right) \qquad (2)$$

To avoid the need to work with various units (i.e. degrees, KGs,…), for the remainder of this paper it is assumed (with no loss of generality) that $\sigma=1$. This means that all values are measured in units of σ. Therefore equation (2) can be simplified to be:

$$P(Y=y)=\Phi((y+0.5)h-\mu)-\Phi((y-0.5)-\mu) \qquad (3)$$

where both h and μ are measured in units of σ. And y is an integer number.

Theoretically, Y can have an infinite range of values. However, as pointed out by previous researchers [30, 16] all values other than the central 5 are negligible. Thus:

$$Y \in \{h_0 - 2h, h_0 - h, h_0, h_0 + h, h_0 + 2h\} \quad (4)$$

Where h_0 is the mode (note that, unlike for the original distribution X, for the observed distribution the mean is not necessarily equal to the mode). The relationship between the original and observed distributions are depicted in **Fig. 1**

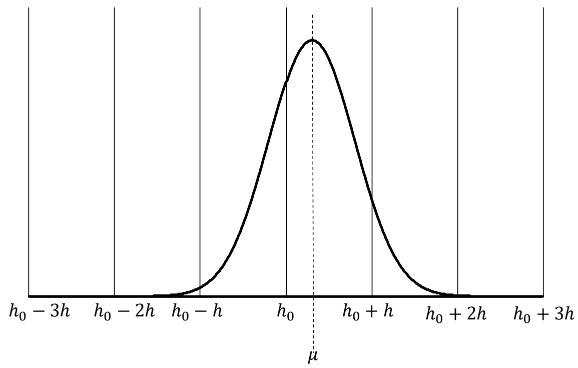

Fig. 1 - Relationship between the X probabilty distribution fuction and the Y probability

To increase accuracy, in the remainder of this paper, it was decided to use 7 values of Y (instead of Gertsbakh proposal of 5 [16]).

$$Y \in \{h_0 - 3h, h_0 - 2h, h_0 - h, h_0, h_0 + h, h_0 + 2h, h_0 + 3h\} \quad (5)$$

These added values help in coping with non-centered distribution ad demonstrated in

Fig. 2 - Relationship between X and Y: (a) Centered case. (b) Deviation from the center

The original upper and lower control limits are set so the alarm sets off when either of the following conditions is met:

$$\bar{X} \leq LCL \quad or \quad \bar{X} \geq UCL \quad (6)$$

Since $\bar{X}$ cannot be measured, there is a need to find a replacement for (6) that depends on the calculated value of $\bar{Y}$.

Similar to the values of Y, the values of $\bar{Y}$ are:

$$\bar{Y} \in \left\{h_0 - 3h, h_0 - \left(3 - \frac{1}{n}\right)h, h_0 - \left(3 - \frac{2}{n}\right)h, \ldots, h_0, h_0 + \frac{h}{n}, \ldots, h_0 + 3h\right\} \quad (7)$$

To calculate the distribution function of $\bar{Y}$ let us define an auxiliary set of variables:

n_0 – number of $y_i{}'$s equal h_0. n_{-1} – number of $y_i{}'$s equal $h_0 - h$. n_{-2} – number of $y_i{}'$s equal $h_0 - 2h$. n_{-3} – number of $y_i{}'$s equal $h_0 - 3h$. n_1 – number of $y_i{}'$s equal $h_0 + h$. n_2 – number of $y_i{}'$s equal $h_0 + 2h$. n_3 – number of $y_i{}'$s equal $h_0 + 3h$.

From these definitions we can deduce that the total number is:

$$n = \sum_{i=-3}^{3} n_i \qquad (8)$$

And also calculate the distribution function of $\bar{Y}$:

$$P(\bar{Y} = v) = P\left(\sum_{i=-3}^{3} i \cdot n_i = nv \middle| \sum_{i=-3}^{3} n_i = n\right) \qquad (9)$$

Because of the multinomial distribution of $\bar{Y}$:

$$P(\bar{Y} = v) = \left[\sum_{i=-3}^{3} \frac{n!}{\prod_{i=-3}^{3} n_i!} \prod_{i=-3}^{3} p_i^{n_i} \middle| \sum_{i=-3}^{3} i \cdot n_i = nv, \sum_{i=-3}^{3} n_i = n\right] \qquad (10)$$

where $p_i = P(Y = h_0 + ih)$

4. Need for Setting new UCL and LCL

The original setting (for "conventional", exact values) of the control limits was based on the requirement for a certain level of false alarms. Shewhart [1] set the level to $\Phi(-3) + (1 - \Phi(3)) = 0.27\%$. This value can be translated to an average run length (ARL) of 370 samples before false alarm. This calculation is not valid for rounded-off measurements (assuming we use the original values). The actual ARL depends on the "roughness" of the rounding (defined as δ, where δ is $\frac{1}{h}$). The actual ARL is depicted in **Fig. 3**.

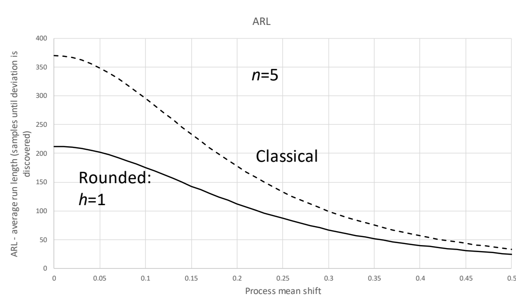

Fig. 3 - ARL for the case of sample size=5 and h=1

From **Fig. 3** it is clear that to maintain the requirements of the $\bar{X}$-chart, a new method to establish the control limits is required.

5. Asymmetry

Due to the characteristics of the normal distribution, the original control limits were symmetrical around the mean (and mode). This symmetry is not preserved as demonstrated in **Fig. 1**, since the measurement device is typically set to arbitrary human-related scale steps (e.g. rounded to a whole number of degrees when measuring temperature) whereas the actual mean is not constrained by this limitation (e.g. it can be 103.4^{0}C). This asymmetry (η) can be expressed as the deviation of the mean from the mode (in units of σ) as depicted in **Fig. 4**

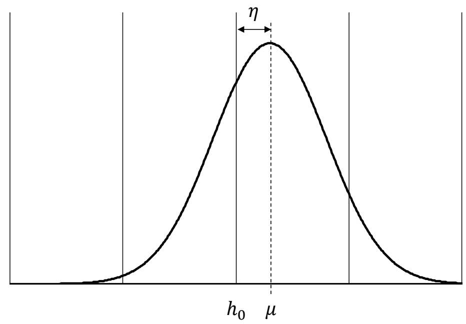

Fig. 4 - Asymmetry (η)

Obviously, the value of η varies in a small range: $-0.5h \leq \eta \leq 0.5h$ as higher/lower values simply indicate a change of the mode. The higher the asymmetry, the higher the probability of false positive alarms:

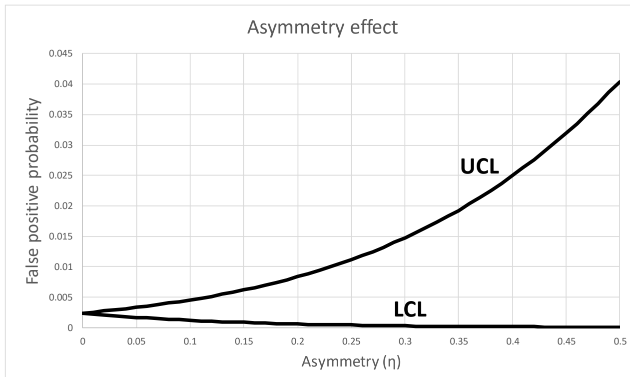

Fig. 5 - Probabilities of getting false positive

The conclusion from this section is that when designing UCL and LCL the asymmetry should be considered.

6. Setting control limits

As decided, the assumption is that there is a need to maintain the commonly used ARL [32, 33, 34, 35, 36] of about 370. This value is also divided equally between the UCL and the LCL (i.e., both have an ARL value of about 741).

The new UCL/LCL should replace (6) and therefore:

$$\bar{Y} < LCL \quad or \quad \bar{Y} > UCL \qquad (11)$$

To maintain the common requirements of ARL, it can be deduced that:

$$P(\bar{Y} > UCL) \geq 0.00135 \qquad (12)$$

$$P(\bar{Y} < LCL) \geq 0.00135$$

To achieve these goals, the following steps should be performed:

- Convert the units to σ units
- Calculate the distribution of $\bar{Y}$
- Using the distribution, set UCL so as $P(\bar{Y} > UCL) \geq 0.00135$
- Set LCL so as $P(\bar{Y} < LCL) \geq 0.00135$

An example of the process is depicted in the 2 following sub-sections.

A. Calculating $\bar{Y}$ distribution

Let us examine an arbitrary case where the sample size (n) is 3 and the mean (μ) of the process is 101.2 and the standard deviation (σ) is 8. Also, let us assume that the measuring device is capable of measuring in multiplications of 10.

From these data we can calculate the following parameters:

$$h = \frac{10}{8} = 1.25$$

$$\delta = \frac{1}{h} = 0.8$$

$$\eta = \frac{\mu - h_0}{\sigma} = \frac{101.2 - 100}{8} = 0.15$$

Using (5) we can deduce that $Y \in \{-3.75, -2.5, -1.25, 0, 1.25, 2.5, 3.75\}$ and the distributions is depicted in **Table 1**.

Table 1 - Distribution of Y

Value	Probability
-3.75	$\Phi\left(\frac{-2.5 \cdot 1.25 - 0.15}{1}\right) - \Phi\left(\frac{-3.5 \cdot 1.25 - 0.15}{1}\right) \cong 0.000526$
-2.5	$\Phi\left(\frac{-1.5 \cdot 1.25 - 0.15}{1}\right) - \Phi\left(\frac{-2.5 \cdot 1.25 - 0.15}{1}\right) \cong 0.0209$
-1.25	$\Phi\left(\frac{-0.5 \cdot 1.25 - 0.15}{1}\right) - \Phi\left(\frac{-1.5 \cdot 1.25 - 0.15}{1}\right) \cong 0.1977$
0	$\Phi\left(\frac{0.5 \cdot 1.25 - 0.15}{1}\right) - \Phi\left(\frac{-0.5 \cdot 1.25 - 0.15}{1}\right) \cong 0.4634$
1.25	$\Phi\left(\frac{1.5 \cdot 1.25 - 0.15}{1}\right) - \Phi\left(\frac{0.5 \cdot 1.25 - 0.15}{1}\right) \cong 0.2751$
2.5	$\Phi\left(\frac{2.5 \cdot 1.25 - 0.15}{1}\right) - \Phi\left(\frac{1.5 \cdot 1.25 - 0.15}{1}\right) \cong 0.0408$
3.75	$\Phi\left(\frac{2.5 \cdot 1.25 - 0.15}{1}\right) - \Phi\left(\frac{1.5 \cdot 1.25 - 0.15}{1}\right) \cong 0.0015$

For each possible value of $\bar{Y}$ its probability can be calculated. For example, to calculate $P(\bar{Y} = -2.5)$ there is a need to map all possibilities of achieving this value. There are 3 cases for this value, as depicted in **Table 2**.

Table 2 - Calculation of $\bar{Y}$

Probability	Case
9.136E-06	All 3 measured values equal to -2.5
1.31E-05	One measured value equals to -3.75, one to -2.5 and one to -1.25
3.881E-07	2 measured values equal to -3.75 and one equals to 0.

The same needs to be calculated for each possible value of $\bar{Y}$, yielding the results depicted in **Table 3**.

Table 3 - $\bar{Y}$ distribution

$\bar{Y}$	Probability
-3.75	1.47457E-10
-3.33333	1.75048E-08
-2.91667	8.5824E-07
-2.5	2.26278E-05
-2.08333	0.000352164
-1.66667	0.003368534
-1.25	0.020102063
-0.83333	0.075137545
-0.41667	0.176865788
0	0.262787019
0.416667	0.246276764
0.833333	0.145807035
1.25	0.054455678
1.666667	0.012770963
2.083333	0.001874032
2.5	0.000169559
2.916667	9.08671E-06
3.333333	2.6267E-07
3.75	3.14385E-09

B. Setting control limits

From **Table 3** the calculation of the UCL and LCL is quite straight forward.

Let us find the lowest value of $\bar{Y}$ that satisfies the requirements of the ARL:

$$P\left(\bar{Y} > 5/3\right) = 0.002053 > 0.00135$$

Similarly, let us find the LCL:

$$P\left(\bar{Y} < -1.25\right) = 0.00374 > 0.00135$$

The final step is to convert back to the original units, that is:

$$UCL = h \cdot UCL \cdot \sigma + h_0 = 1.25 \cdot \frac{5}{3} \cdot 8 + 100 = 116.67$$

7. Conclusion

The proposed method provides a simple and effective tool to set control limits for various statistical process controls. The technique maintains the original characteristics of the "conventional" limits while coping with the problem of rounded-off data.

Similar control limits should be developed for the more complex cases of R-charts and other control techniques.

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DOI: https://doi.org/10.15379/ijmst.v10i4.2249