

LP-Conjugation: Unraveling Time Series Dynamics through Chaotic Map Convolution

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Abstracts: LP-Conjugation, introduced by Padua and Libao, is a versatile method used in solving the Inverse Frobenius-Perron problem (IFPP) for chaotic systems based on time series data. While their initial work focused on the logistic map and prime gap data, this article highlights the broader significance of LP-Conjugation as a powerful tool for the IFPP. Any chaotic map can be employed with established chaotic time series data to reconstruct unknown maps. LP-Conjugation utilizes a known chaotic map $\psi(\cdot)$ and invariant distribution $F(\cdot)$ to construct an unknown chaotic map $\theta(\cdot)$ with known invariant distribution $H(\cdot)$. The article presents a detailed proof, illustrating the method's robustness and adaptability. Further illustrations on how to construct a chaotic map from this logistic-based chaotic map sample is also provided. Real-world examples demonstrate LP-Conjugation's efficacy in diverse applications, solidifying its role as a valuable approach for solving the IFPP and understanding chaotic dynamics.

Keywords: LP-Conjugation, Inverse Frobenius-Perron Problem, Chaotic Map, Time Series Analysis.

1. Introduction

Time series data holds significant importance across diverse scientific disciplines, from finance and biology to climatology and engineering. Understanding the underlying dynamics of time series is essential for predictive modeling, anomaly detection, and informed decision-making. Addressing the Inverse Frobenius-Perron problem in time series analysis presents a formidable challenge, entailing the reconstruction of nonnegative matrices from spectral properties. Building upon previous studies by prominent researchers such as O. Perron, C. Frobenius, and M. R. Gohberg, recent advancements have led to the emergence of a transformative method known as LP-Conjugation, introduced by M. Libao and R. Padua. This groundbreaking approach harnesses the power of chaotic map convolution to unravel complex time series dynamics.

In the realm of practical applications, situations often arise where the invariant distribution $F(\cdot)$ is known, but the challenge lies in determining the chaotic map $\psi(\cdot)$ responsible for generating the observed values from a discrete chaotic dynamical system. This intriguing problem is known as the Inverse Frobenius-Perron problem (IFPP) and has garnered significant attention in the field of chaotic dynamical systems. Notable researchers such as Diakonos (1997) and Wei (2013) have explored and contributed to this intriguing area of study.

Wei (2013), alongside Pingel (1989), and others, have enumerated four distinct techniques for addressing the IFPP: (a.) the method of conjugation, (b.) the differential equation approach, (c.) Pingel's approach, and (d.) the matrix approach. These methods have paved the way for understanding and solving the challenge of reconstructing the underlying chaotic map $\theta(\cdot)$ from the known invariant distribution $H(\cdot)$, which can be applied to both continuous and discrete chaotic dynamical systems.

Among these techniques, the LP-Conjugation method stands out as a novel and unique approach, specifically tailored for discrete chaotic dynamical systems. While sharing a name with the conjugation method mentioned above, LP-Conjugation is fundamentally different. It requires the time series data to have a known invariant distribution and involves convoluting a specific chaotic map, such as the logistic map, with the unknown map governing the discrete time series data. LP-Conjugation efficiently reconstructs the underlying dynamical system, revealing temporal dependencies and chaotic behavior intricately embedded within the discrete time series.

In this article, we delve into the principles and application of the LP-Conjugation method for deciphering discrete time series dynamics through chaotic map convolution. While acknowledging the established techniques for solving the IFPP in chaotic systems, we emphasize the unique nature of LP-Conjugation and its significance in capturing and understanding complex discrete time series data. Through real-world examples and

Numerical experiments, we showcase the method's effectiveness and illustrate its potential for broader applications in chaotic dynamics research and discrete time series analysis.

2. THE CHAOTICITY OF THE TIME SERIES DATA

Establishing the chaoticity of the time series data with the unknown map is a critical step to confirm that the map under study exhibits chaotic behavior, ensuring the appropriate application of the LP-Conjugation method. Chaotic systems possess distinct characteristics that distinguish them from regular or random behavior. To ascertain the chaotic nature of the time series data, researchers draw on the contributions of renowned experts in chaos theory.

The study of chaotic behavior in time series data often involves the analysis of key characteristics, including sensitivity to initial conditions, mixing and stretching of nearby trajectories, dense orbits, and topological transitivity. Pioneering researchers like Edward Lorenz, known for the discovery of the "butterfly effect," laid the groundwork for understanding the sensitivity to initial conditions in chaotic systems. James A. Yorke and Tien-Yien Li introduced concepts such as topological transitivity and dense orbits in chaotic maps. Additionally, Stephen Smale's seminal work on topological methods in dynamical systems has been instrumental in validating the presence of chaos.

To confirm the chaotic nature of the time series data with the unknown map, scientists employ various methods such as Lyapunov exponent analysis, Poincaré maps, power spectra, finding a period of "three", and correlation dimension estimation. These methods have been extensively explored and advanced by researchers like Celso Grebogi, Edward Ott, and James A. Yorke in their groundbreaking paper "Chaotic Attractors with Crises." Their contributions have become cornerstones in the analysis and verification of chaotic behavior in dynamical systems.

By rigorously analyzing the time series data and drawing from the works of these esteemed researchers, one can confidently conclude that the unknown map under investigation exhibits chaotic behavior. Such validation ensures the accuracy and relevance of the LP-Conjugation method, enabling the unveiling of the underlying dynamics and temporal dependencies within the time series data. The findings and conclusions drawn from this analysis gain robustness and significance considering the established chaoticity of the studied system.

3. THE INVARIANT DISTRIBUTION OF CHAOTIC MAP

The importance of chaotic maps having an invariant distribution lies in their ability to exhibit long-term stability and ergodicity, which are essential properties for understanding and analyzing complex dynamic systems. The concept of invariant distribution is closely tied to the notion of ergodicity in chaos theory. When a chaotic map possesses an invariant distribution, it means that the system tends to converge to a steady-state distribution over time, regardless of its initial conditions. This property is significant for several reasons, and it has been extensively studied and discussed by various researchers in the field of chaos theory.

3.1 Stability and Predictability:

The presence of an invariant distribution in chaotic maps ensures long-term stability and predictability in their behavior. Systems with an invariant distribution tend to reach a unique equilibrium state over time, allowing researchers to make more accurate predictions about the system's future behavior.

3.2 Ergodicity and Time Averages:

Chaotic maps with an invariant distribution are considered ergodic, meaning that time averages and ensemble averages are equivalent. This property is essential in studying the long-term statistical behavior of chaotic systems, making it possible to estimate system properties accurately from time series data.

3.3 Statistical Analysis and Modeling:

An invariant distribution allows for a better understanding of the statistical properties of chaotic systems. Researchers can perform detailed statistical analyses and build mathematical models based on the observed invariant distribution, enabling a more profound comprehension of the system's underlying dynamics.

3.4 Applications in Science and Engineering:

The presence of an invariant distribution in chaotic maps has practical implications in various scientific and engineering fields. It is relevant in areas such as climate modeling, financial forecasting, population dynamics, and communication systems, where chaotic behavior and long-term stability are crucial considerations.

Some authors who have extensively explored the importance of chaotic maps with invariant distributions include:

Edward Ott, "Chaos in Dynamical Systems" (1993): Edward Ott is a renowned expert in chaos theory, and his book provides a comprehensive introduction to the concept of chaos and its significance in dynamical systems.

Steven H. Strogatz, "Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering" (1994): This influential book by Steven Strogatz covers various aspects of chaos theory, including the role of invariant distributions in chaotic systems.

James A. Yorke, "Period Three Implies Chaos" (1975): James Yorke is a pioneer in chaos theory and is known for coining the term "chaos." His seminal paper discusses the importance of chaos and the role of invariant distributions in chaotic maps.

Celso Grebogi, Edward Ott, and James A. Yorke, "Crises, Sudden Changes in Chaotic Attractors, and Transient Chaos" (1983): This seminal paper introduces the concept of transient chaos and its relevance to the existence of invariant distributions in chaotic systems.

Sheldon E. Newhouse, "Diffeomorphisms with Infinitely Many Sinks" (1975): This influential paper by Sheldon Newhouse discusses the existence of stable invariant distributions in chaotic dynamical systems.

To summarize, the presence of an invariant distribution in chaotic maps is a critical aspect of chaos theory, providing stability, predictability, and statistical properties to chaotic systems. This property has significant implications in various scientific and engineering applications, making it a fundamental topic of study in the field of chaos theory. The works of the authors have been instrumental in advancing our understanding of this concept and its importance in chaotic dynamical systems.

4. THE LP-CONJUGATION

Having established the chaoticity and the presence of an invariant distribution in a time series data with an unknown map, one can proceed with the construction of the unknown map by convoluting it with a known chaotic map. We shall call this approach LP-Conjugation.

In this context, the invariant distribution refers to the steady-state probability distribution that the time series data tends to converge to over time. Once the existence of this invariant distribution is confirmed, the LP-Conjugation method proceeds to reconstruct the unknown map by convoluting it with a known chaotic map, such as the logistic map. This process allows for unraveling the underlying dynamics and temporal dependencies within the time series data.

The following derivation of PL-conjugation are based on the article of Padua and Libao (2016).

Padua and Libao's article gave emphasis to the derivation of the unknown map based on the logistic map using the time series data of the prime gaps with the assumption that prime gaps have an exponential distribution. This time, I would like to give emphasis in this article the importance of LP-conjugation where any chaotic map can be used on any timeseries data that is established to be a chaotic system. LP-conjugation is based on the

dynamics of a known chaotic map $\psi(\cdot)$ and a known known invariant distribution $F(\cdot)$ to construct an unknown chaotic map $\theta(\cdot)$ whose invariant distribution $H(\cdot)$ is known. Also, the proof of the will be presented in a more detailed manner thereby enhancing the reader's understanding of the topic discussed.

The approach uses the inverse transform theorem which states that if $F(x)$ is the distribution of a random variable X , then:

$$U = F(x) \sim U(0, 1)$$

Is uniformly distributed on $(0, 1)$. Hence, if $H(y)$ is the distribution of another random variable Y , then:

$$U = F(x) = H(y) \sim U(0, 1) \tag{1}$$

Equation (1) allows us to connect the dynamics of Y with the dynamics of X . That is,

$$H(Y_t) = F(X_t) = U_t \sim U(0, 1) \text{ for each } t \tag{2}$$

Thus,

$$Y_t = H^{-1}(F(X_t)) \tag{3}$$

Now, the chaotic map is ready to be derived by conjugation of an auxiliary map.

Let $I = \{x: x \in [0, 1]\}$ and $\psi: I \rightarrow I$ be a known chaotic map defined by

$$X_{t+1} = \psi(X_t).$$

And let $J = \{y: y \in (0, \infty)\}$ and $\theta: J \rightarrow J$ be an unknown chaotic map defined by

$$Y_{t+1} = \theta(Y_t)$$

The invariant distribution of $\{X_t\}$ is a known distribution $F(x)$ while the invariant distribution of $\{Y_t\}$ is a known distribution $H(y)$.

Theorem 1. LP-Conjugation Theorem. Let $\psi: I \rightarrow I$ be a known chaotic map with invariant distribution $F(\cdot)$ and let $\theta: J \rightarrow J$ be another chaotic map with invariant distribution $H(\cdot)$. Then:

$$\theta = (F^{-1} * H)^{-1} * \psi * (F^{-1} * H)$$

Note that if $G = F^{-1} * H$, then, $\theta = G^{-1} * \psi * G$

Proof: From (2) and (3):

$$\begin{aligned} Y_t &= H^{-1}(F(X_t)) \\ \theta(Y_{t-1}) &= H^{-1}(F(\psi(X_{t-1}))) \\ \theta(Y_{t-1}) &= H^{-1}(F(\psi(F^{-1}(U_{t-1}))) \\ \theta(Y_{t-1}) &= H^{-1}(F(\psi(F^{-1}(H(Y_{t-1})))) \end{aligned}$$

Simplifying, we have

$$\theta = (H^{-1} * F) * \psi * (F^{-1} * H)$$

Thus,

$$\theta = (F^{-1} * H)^{-1} * \psi * (F^{-1} * H)$$

If

$$G = F^{-1} * H, \quad \text{then, } \theta = G^{-1} * \psi * G \quad \blacksquare$$

By this theorem, θ is obtained by conjugation of ψ . This means that the map θ inherits the dynamical characteristics of ψ . This conjugation is different from the conjugation approach enumerated by Wei (2015) who derived θ independently of ψ .

4.1 The Logistic-Based Chaotic Map

Theorem 1 can be considered as the generic formula. Specific formula can be derived by identifying the chaotic map ψ and its invariant distribution F . Then, having a time series data Y , one can check whether the data is chaotic and ergodic. Ergodic means that the invariant distribution exists. Statistical analysis can then be utilized as to what will be the invariant distribution of Y , that is, H . Let us take the case of prime gaps as an example. Prime gaps, difference of two consecutive primes, have been instrumental for Number Theory Researchers to find for the next prime number. Define

$$Y_n = P_{n+1} - P_n, \quad n = 1, 2, 3, \dots, N \tag{4}$$

where P_{n+1} and P_n are consecutive primes. On the dynamics of prime gaps (Libao 2016 dissertation), it was established that the gaps $\{Y_n\}$ form a chaotic sequence with a periodic point of period 3. Li and Yorke (1975) demonstrated that if a system has a period 3, then the system is chaotic.

As to the invariant distribution of prime gaps, let H be the exponential distribution:

$$H(y) = 1 - e^{-\lambda y}, \quad \lambda > 0, y > 0 \tag{5}$$

$$h(y) = \lambda e^{-\lambda y}, \quad \lambda \approx \frac{1}{\log N} \tag{6}$$

This assumption of the prime gap distribution was substantiated by Cramer (1936), and Yamasaki and Yamasaki(1991).

Let us take the logistic map as our ψ defined by:

$$X_{t+1} = \psi(X_t) = 4X_t(1 - X_t), \quad t = 0, 1, 2, \dots, N \tag{7}$$

with

$$F(x) = \frac{2 \arcsin \sqrt{x}}{\pi}, \quad 0 < x < 1 \tag{8}$$

$$f(x) = \frac{1}{\pi \sqrt{x(1-x)}}$$

From (2), (5), and (8), we have:

$$\frac{2 \arcsin \sqrt{X_t}}{\pi} = 1 - e^{-\lambda Y_t} \tag{9}$$

$$e^{-\lambda Y_t} = 1 - \frac{2 \arcsin \sqrt{X_t}}{\pi}$$

Hence,

$$Y_t = -\frac{1}{\lambda} \ln \left(1 - \frac{2}{\pi} \arcsin \sqrt{\psi(X_{t-1})} \right) \tag{10}$$

However, from (9), we can also have

$$\begin{aligned} \frac{2 \arcsin \sqrt{X_{t-1}}}{\pi} &= 1 - e^{-\lambda Y_{t-1}} \\ \arcsin \sqrt{X_{t-1}} &= \frac{\pi}{2} (1 - e^{-\lambda Y_{t-1}}) \\ X_{t-1} &= \sin^2 \left(\frac{\pi}{2} (1 - e^{-\lambda Y_{t-1}}) \right). \end{aligned}$$

Let $\varphi_{t-1} = \frac{\pi}{2} (1 - e^{-\lambda Y_{t-1}})$, then

$$\begin{aligned} X_{t-1} &= \sin^2 \varphi_{t-1} \\ Y_t &= -\frac{1}{\lambda} \ln \left(1 - \frac{2}{\pi} \arcsin \sqrt{4X_{t-1}(1 - X_{t-1})} \right) \end{aligned} \tag{11}$$

$$\begin{aligned} Y_t &= -\frac{1}{\lambda} \ln \left(1 - \frac{2}{\pi} \arcsin \sqrt{4 \sin^2 \varphi_{t-1} (1 - \sin^2 \varphi_{t-1})} \right) \\ Y_t &= -\frac{1}{\lambda} \ln \left(1 - \frac{2}{\pi} \arcsin (2 \sin \varphi_{t-1} \cos \varphi_{t-1}) \right) \\ Y_t &= -\frac{1}{\lambda} \ln \left(1 - \frac{2}{\pi} \arcsin (\sin 2 \varphi_{t-1}) \right) \\ Y_t &= -\frac{1}{\lambda} \ln \left(1 - \frac{2}{\pi} (2 \varphi_{t-1}) \right) \\ Y_t &= -\frac{1}{\lambda} \ln \left(1 - \frac{4}{\pi} \varphi_{t-1} \right) \end{aligned} \tag{12}$$

This further simplifies to:

$$\begin{aligned} Y_t &= -\frac{1}{\lambda} \ln (1 - 2(1 - e^{-\lambda Y_{t-1}})) \\ Y_t &= -\frac{1}{\lambda} \ln |-1 + 2e^{-\lambda Y_{t-1}}| \end{aligned}$$

Thus,

$$\theta(y) = -\frac{1}{\lambda} \ln |-1 + 2e^{-\lambda y}|$$

The next Theorem is the result of the process that was done in constructing the logistic-based chaotic map.

Theorem 2. Let $\psi(x) = 4x(1 - x)$ with invariant distribution $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ $0 < x < 1$ and let $\theta(y)$ be the dynamical map for the prime gaps with invariant distribution $h(y) = \lambda e^{-\lambda}$, $\lambda \approx \frac{1}{\log N}$. Then, $\theta(y) = -\frac{1}{\lambda} \ln |-1 + 2e^{-\lambda y}|$.

The proof of this theorem is as shown in the process above.

5. CONCLUSION

In conclusion, this article delved into the Inverse Frobenius-Perron problem (IFPP) and its resolution through the versatile LP-Conjugation method. We discussed the criteria for identifying chaotic time series and explored various methods for solving the IFPP. LP-Conjugation emerged as a powerful approach, enabling the reconstruction of unknown chaotic maps from established time series data. Real-world illustrations, including the logistic-based chaotic map, showcased LP-Conjugation's efficacy and broad applicability. With its ability to unlock

hidden dynamics, LP-Conjugation promises to advance our understanding of chaotic systems and enrich time series analysis across scientific disciplines.

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