

# Numerical Solution of Complex Fuzzy Differential Equations by Euler and Taylor Methods

Mayada Abualhomos\*

*Basic Science, Faculty of Art and Science, Applied Science Private University, Amman, 11193, Jordan; E-mail: [abuhomos@asu.edu.jo](mailto:abuhomos@asu.edu.jo)*

**Abstracts:** In 1965, Zadeh introduced the concept of fuzzy set, which is a class of objects with a continuum of grades of membership, such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. In 1987 O. Kaleva defined the concept of fuzzy differential equations and present some basic notions of differential equations such as differentiability, integrability, existence and uniqueness theorem for a solution to a fuzzy differential equation. He also, in 1990 studied the Cauchy problem for fuzzy differential equations and showed that it has a solution if and only if there is a subset and its locally compact. Later, M. Ma, M. Friedman, and A. Kandel 1999 introduced numerical solutions of fuzzy differential equations. In this paper we incorporate the above ideas to introduce numerical solution of complex fuzzy differential equations by Euler and Taylor methods by extending the codomain of membership function of fuzzy topological space from  $[0, 1]$  to the unit disk in the complex plane. This extension allows us getting more range and flexibility to represent objects with uncertainty and periodicity semantics without losing the full meaning of information. Also, we considered the definitions of complex fuzzy sets, cartesian and polar representation of complex membership, and Cauchy problem for CFDEs. We then found the exact solutions and approximations for Taylor and Euler methods for CFDEs by  $\alpha$  -levels and  $\theta$  where  $0 \leq \alpha, \theta \leq 1$ , and provide examples of the results we obtained.

**Keywords:** Complex Fuzzy Sets, Complex Fuzzy Differential Equations, Cauchy Problem.

## 1. INTRODUCTION

In 1965 Zadeh [8] was interduce the concept of fuzzy set. The first order Cauchy problem is:

$$x'(t) = f(t, x(t)), x(t_0) = x_0$$

has a solution if  $f$  is continuous. A fuzzy differential equation has been studied extensively like Peano's theorems see [9]. The idea of extending the range values from  $[0, 1]$  to unit disk in the complex plane has successfully spread in different areas and has useful applications in the real life. So, in (2002,2003) Ramot et. al. [10,11] proposed this idea by introducing the concept of complex fuzzy set and logic, he extended the range of membership function to a unit disk in the complex plane. Hence, for translating some complex-valued functions on physical terms to human language and vice versa he added phase term to solve this enigma. In (2012) Jun et al. [7] used complex fuzzy sets to represent the information with the uncertainty and periodicity simultaneously, where they generate a solution of multiple periodic factor prediction (MPFP) problems. In (2012) Zhifei et al. [18], the concept of complex fuzzy logic has been applied in a neuro-fuzzy system architecture. Many other researchers who combined, generalized and applied the concept of complex fuzzy sets, as, Alkouri and Salleh [2-4], Zhang et al. [17], Tamir and Kandel [13] and Tamir et al. [14-16]. Later, Alhusban and Salleh [1] incorporate the above ideas to introduce complex fuzzy space and applied it in complex fuzzy group theory. Tamir et al. [6], interduce a new interpretation of complex membership grade, and presented cartesian and polar representation of complex grades of membership. Karpenko et al. [5] based on pure complex fuzzy sets they studied the existence of a solution for the Cauchy problem for FDEs. Later, S. Abbasbandy, T.A. Viranloo [12], used Taylor method to solve Numerical solutions of fuzzy differential equation. In this paper we will using cartesian and polar representation of complex grades of membership to define Taylor and Euler methods for complex fuzzy differential equations.

## 2. LITERATURE REVIEW

In this section, we introduce basic, definitions, theorems and notations which are used throughout this paper.

**Definition 2.1 [11]** A complex fuzzy set  $S$ , defined on universe of discourse  $U$ , is characterized by membership function  $\mu_s(x)$ ,

$$\mu_s(x) = r_s(x) \cdot e^{j\omega_s(x)}$$

where  $j = \sqrt{-1}$ ,  $r_s(x)$  and  $\omega_s(x)$  are both real-valued, and  $0 \leq r_s(x) \leq 1$ .

The complex fuzzy set  $S$  it's an ordered pair.

$$S = \{(x, \mu_s(x)) \mid x \in U\}.$$

At the first, will denote  $\mathcal{P}_k(\mathbb{R}^n)$  for the set of all nonempty compact convex subsets of  $\mathbb{R}^n$ .

An expansion of the original definition in [11] referred as “pure complex fuzzy” in [6], where the cartesian and a polar representation of complex membership are defined in Tamir et al. At the first will consider the cartesian definition.

The concept of cartesian and polar representation of complex membership was introduced in [6]

### 2. 1. Cartesian representation of complex membership

In [6] defined the complex membership functions  $\mu$  as:

$$\mu(G, s) = \mu_R(G) + i\mu_I(s)$$

where  $G$  is a fuzzy set and  $s$  belong to  $G$ . We can extend this definition to  $\mathbb{R}^n$  by make  $x \in \mathbb{R}^n$ .

$$f(x) = u(x) + iv(x)$$

Where  $f(u, v): \mathbb{R}^n \rightarrow [0,1]$ ,  $f$  assigns to each  $x \in \mathbb{R}^n$  a value in the unit square in complex plane  $\mathbb{C}$ , representing a complex grade of membership.  $u, v$  expresses non-complex fuzzy set in  $\mathbb{R}^n$ .

Now, for  $u: \mathbb{R}^n \rightarrow [0,1]$ ,  $\alpha$  –level sets are defined as:

$$[u]^\alpha = \{x \in \mathbb{R}^n \mid u(x) \geq \alpha, \quad \alpha \in (0, 1]\},$$

$$[u]^0 = \overline{\{x \in \mathbb{R}^n \mid u(x) \geq 0\}}.$$

Now, we can define  $(\alpha, \beta)$  –level where  $\alpha, \beta \in (0,1]$ , for  $f$  as:

$$[f]^{(\alpha,\beta)} = [u]^\alpha \cap [v]^\beta. \tag{2.1}$$

The following set are conditions as an alternative definition of  $[f]^{(\alpha,\beta)}$ :

$$[f]^{(\alpha,\beta)} = \{x \in \mathbb{R}^n \mid u(x) \geq \alpha, v(x) \geq \beta, \alpha, \beta \in (0, 1]\}, \tag{2.2}$$

$$[f]^{(\alpha,0)} = \overline{\{x \in \mathbb{R}^n \mid u(x) \geq \alpha, v(x) \geq 0, \alpha \in (0, 1]\}}, \tag{2.3}$$

$$[f]^{(0,\beta)} = \overline{\{x \in \mathbb{R}^n \mid u(x) \geq 0, v(x) \geq \beta, \beta \in (0, 1]\}}, \tag{2.4}$$

$$[f]^{(0,0)} = \overline{\{x \in \mathbb{R}^n \mid u(x) \geq 0, v(x) \geq 0\}}, \tag{2.5}$$

Notice that (2.2) and (2.5) are equivalent to (2.1) for the corresponding  $\alpha, \beta$ , but that (2.3) and (2.4) are not equivalent to (2.1) and may not generate closed sets when one of  $\alpha, \beta$  is equal 0, but (2.3) and (2.4) would generate the respective closures of those sets.

Now, let  $E^n$  be the set of all  $m: \mathbb{R}^n \rightarrow [0,1]$ , we have  $[u]^\alpha \cap [v]^\beta$  is always compact and convex for  $u, v \in E^n$ , and  $[u]^1 \cap [v]^1 \subset [u]^\alpha \cap [v]^\beta \subset [u]^0 \cap [v]^0, \forall \alpha, \beta \in [0,1]$ , for make sure that  $[u]^\alpha \cap [v]^\beta$  be nonempty and  $[u]^1 \cap [v]^1$  be nonempty, therefore there exist  $x_0 \in \mathbb{R}^n$ , such that  $x_0 \in [u]^1$  and  $x_0 \in [v]^1$ . Therefore, we define the following set:

$$\hat{E}^{2n} = \{(u, v) \in E^n \times E^n | \exists x_0 \in \mathbb{R}^n \text{ s.t } u(x_0) = v(x_0) = 1\}. \tag{2.6}$$

Then  $f \in \hat{E}^{2n}$ ,  $[f]^{(\alpha, \beta)} \in \mathcal{P}_k(\mathbb{R}^n), \forall (\alpha, \beta) \in [0, 1]$ , and the compactness of the  $[f]^{(\alpha, \beta)}$  sets guarantee the complete equivalence of (2.1) and the set of (2.2) – (2.5).

### 2.2. Polar representation of complex membership

In [6] defined the complex membership functions  $\mu$  as:

$$\mu(G, s) = r(G)e^{j\omega\theta(s)}$$

where  $r(G)$  and  $\omega$  are both real-valued, does not translate quickly to and from the particular cartesian representation. Hence, the two representations of the corresponding extension to  $\mathbb{R}^n$  are not equivalent.

Now, the polar form is:

$$f(x) = r(x)e^{2\pi j\theta(x)},$$

Where  $f(r, \theta): \mathbb{R}^n \rightarrow [0, 1]$ , and  $2\pi$  is taken scaling factor, allowing the range of  $f$  to be the unit circle. The value of  $\theta$  giving by the maximum distance from  $e^0, \theta = 0.5$ , to be the “maximum” membership value because  $e^{2\pi j\theta}$  is periodic. Now, we can define  $[r]^\alpha$  as  $[u]^\alpha$ , the level sets for  $\theta$  denoted by  $[\theta]^{(\beta)}$  defined as:

$$[\theta]^{(\beta)} = \{x \in \mathbb{R}^n: \theta(x) \in [\beta, 1 - \beta], \beta \in (0, 0.5]\},$$

$$[\theta]^{(0)} = \overline{\{x \in \mathbb{R}^n: 0 < \theta(x) < 1\}},$$

$$[\theta]^{(\beta)} = \{[\theta]^{(1-\beta)}, \quad \forall \beta \in [0, 1]\}.$$

Now, we can define  $(\alpha, \beta)$  –level for  $f$  as:

$$[f]^{(\alpha, \beta)} = [r]^\alpha \cap [\theta]^{(\beta)}, \tag{2.7}$$

Or

$$[f]^{(\alpha, \beta)} = \{x \in \mathbb{R}^n: r(x) \geq \alpha, \theta(x) \in [\beta, 1 - \beta]\}. \tag{2.8}$$

$$[f]^{(\alpha, 0)} = \overline{\{x \in \mathbb{R}^n: r(x) \geq \alpha > 0, 0 < \theta(x) < 1\}}, \tag{2.9}$$

$$[f]^{(0, \beta)} = \overline{\{x \in \mathbb{R}^n: r(x) > 0, \theta(x) \in [\beta, 1 - \beta], \beta \in (0, 0.5]\}}, \tag{2.10}$$

$$[f]^{(0, 0)} = \overline{\{x \in \mathbb{R}^n: r(x) > 0, 0 < \theta(x) < 1\}}, \tag{2.11}$$

$$[f]^{(\alpha, \beta)} = [f]^{(\alpha, 1-\beta)}, \forall \alpha, \beta \in [0, 1]. \tag{2.12}$$

It's clearly that  $\theta \in E^n, [\theta]^{(\beta)} \subset [\theta]^\beta, \forall \beta \in [0, 0.5]$ , but  $[\theta]^{(\beta)}$  need not be compact or convex. So, we define  $F^n$  is the set of all  $m$  where  $m: \mathbb{R}^n \rightarrow [0,1]$ , and satisfying the following:

- (i)  $\exists x_0 \in \mathbb{R}^n$  s.t  $m(x_0) = 0.5$ ;
- (ii)  $m$  is monotone,

(iii)  $m$  is upper semi-continuous on  $K_1$  and lower on  $K_2$  where

$$K_1 = \{x \in \mathbb{R}^n | 0 < m(x) \leq 0.5\},$$

$$K_2 = \{x \in \mathbb{R}^n | 0.5 \leq m(x) < 1\}.$$

(iv)  $\overline{K_1 \cup K_2}$  is compact.

Hence, we can define:

$$\hat{E}_*^{2n} = \{(r, \theta) \in E^n \times F^n | \exists x_0 \in \mathbb{R}^n \text{ s.t } r(x_0) = 1 \text{ and } \theta(x_0) = 0.5\}. \tag{2.13}$$

We claim that  $\hat{E}_*^{2n}$  is embeddable into a Banach space, see [15].

### 2.3. The Cauchy problem for complex fuzzy differential equations CFDEs

Let  $\mathcal{E} = \hat{E}^{2n}$  and  $\mathcal{E} = \hat{E}_*^{2n}$  to express the cartesian and polar complex form, respectively. By CFDE, the solution  $X$  is a continuous map  $X: I \rightarrow \mathcal{E}$ . Now, we will use the Hukuhara difference to define the differentiability. For  $x, y \in \mathcal{E}$ , if there exists  $z \in \mathcal{E}$  such that  $x + z = y$ , implies that,  $z = y - x$  and it's called the difference between  $x$  and  $y$ .

**Definition 2.2** The mapping  $F: I \rightarrow \mathcal{E}$  where  $I = [a, b] \subset \mathbb{R}$  be a compact interval, is differentiable at  $t_0 \in I$ , if there exists  $F'(t_0) \in \mathcal{E}$  such that the limits exist and are equal to  $F'(t_0)$ :

$$\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

Now, let  $F: I \rightarrow \mathcal{E}$  and  $H: F: I \rightarrow \mathcal{E} \quad I \rightarrow \mathcal{E}$  are continuous mapping, and we define  $H$  by:

$$H(t) = \int_0^t F(s) ds, \quad t \in I \tag{2.14}$$

When we take the derivative:

$$\frac{d}{dt} H(t) = H'(t) = F(s), \quad t \in I. \tag{2.15}$$

### 2.3. The Cauchy problem for complex fuzzy differential equations CFDEs

Consider the continuous mapping  $G: I \times \mathcal{E} \rightarrow \mathcal{E}$ , and we define the Cauchy problem as:

$$\begin{cases} X'(t) = G(t, X(t)), \quad t \in I \\ X(t_0) = u_{t_0} \end{cases} \tag{2.16}$$

**Definition 2.3** Let  $X: I \rightarrow \mathcal{E}$  is a solution for (2.16) iff  $X$  is continuous and satisfies the following:

$$X(t) = x_a + \int_a^t G(s, X(s)) ds. \tag{2.17}$$

## 3. The Complex Fuzzy Differential Equation

Consider the IVP for complex fuzzy differential equation (2.16):

$$\begin{cases} X'(t) = G(t, X(t)), & t \in I \\ X(t_0) = u_{t_0} \in \mathcal{E} \end{cases}$$

Where  $G: I \times \mathcal{E} \rightarrow \mathcal{E}$  and  $X(t_0)$  is a complex fuzzy number and belong to  $\mathcal{E}$ .

The  $\alpha$  –level denote by:

$$[X(t)]_\alpha = [X_1(t_0; \alpha, \theta), X_2(t_0; \alpha, \theta)]$$

$$[G(t, X(t))]_\alpha = [G_1(t, X(t); \alpha, \theta), G_2(t, X(t); \alpha, \theta)]$$

Where

$$\begin{aligned} G_1(t, X(t); \alpha, \theta) &= \\ \min\{G(t, u) \mid u \in X_1(t_0; \alpha, \theta), X_2(t_0; \alpha, \theta)\}, & \quad (3.1) \\ G_2(t, X(t); \alpha, \theta) &= \\ \max\{G(t, u) \mid u \in X_1(t_0; \alpha, \theta), X_2(t_0; \alpha, \theta)\}. & \end{aligned}$$

The parametric form (3.1) is given by:

$$\begin{aligned} G_1(t, X(t); \alpha, \theta) &= H(t, X_1(t_0; \alpha, \theta), X_2(t_0; \alpha, \theta)), \\ G_2(t, X(t); \alpha, \theta) &= K(t, X_1(t_0; \alpha, \theta), X_2(t_0; \alpha, \theta)). \end{aligned} \quad (3.2)$$

The  $G(t, X)$  is a complex fuzzy process and  $G^{(i)}(t, X)$  is derivative for  $i \in [1, P]$  and denoted by:

$$[G^{(i)}(t, X(t))]_\alpha = [G_1^{(i)}(t, X(t); \alpha, \theta), G_2^{(i)}(t, X(t); \alpha, \theta)], \quad t \in I, \quad (3.3)$$

provided that determines the complex fuzzy number  $G^{(i)}(t, X(t)) \in \mathcal{E}$ , where:

$$\begin{aligned} G_1^{(i)}(t, X(t); \alpha, \theta) &= \\ \min\{G^{(i)}(t, u) \mid u \in X_1(t_0; \alpha, \theta), X_2(t_0; \alpha, \theta)\}, & \quad (3.4) \\ G_2^{(i)}(t, X(t); \alpha, \theta) &= \\ \max\{G^{(i)}(t, u) \mid u \in X_1(t_0; \alpha, \theta), X_2(t_0; \alpha, \theta)\}. & \end{aligned}$$

#### 4. The Euler method

Let the exact solutions:

$$[Y(t_n)]_\alpha = [Y(t_n; \alpha, \theta)] = [Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta)]$$

And

$$[y(t_n)]_\alpha = [y(t_n; \alpha, \theta)] = [y_1(t_n; \alpha, \theta), y_2(t_n; \alpha, \theta)]$$

Be approximation solutions at  $t_n$  where  $n = 0 \dots N$ . The solutions are calculated by grid points at:

$$t_0 < t_1 < t_2 < \dots < t_N = T$$

$$h = \frac{T - t_0}{N}, \quad t_n = t_0 + nh \tag{4.1}$$

The Euler's method is based on first-order approximation of  $Y'_1(t_n; \alpha, \theta)$  and  $Y'_2(t_n; \alpha, \theta)$  and given by:

$$A'(t; \alpha, \theta) = \frac{A(t + h; \alpha, \theta) - A(t; \alpha, \theta)}{h}$$

Where  $A'(t; \alpha, \theta)$  is  $[Y(t_n)]_\alpha$ .

Then we have

$$\begin{aligned} [Y_1(t_{n+1}; \alpha, \theta)] &\approx [Y_1(t_n; \alpha, \theta) + hK(t_n; \alpha, \theta)] \\ [Y_2(t_{n+1}; \alpha, \theta)] &\approx [Y_2(t_n; \alpha, \theta) + hL(t_n; \alpha, \theta)] \end{aligned} \tag{4.2}$$

Where

$$K(t_n; \alpha, \theta) = K(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta))$$

$$L(t_n; \alpha, \theta) = L(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta))$$

From (4.2) we define:

$$\begin{aligned} [y_1(t_{n+1}; \alpha, \theta)] &= \\ [y_1(t_n; \alpha, \theta) + hK(t_n, y_1(t_n; \alpha, \theta), y_2(t_n; \alpha, \theta))] & \\ [y_2(t_{n+1}; \alpha, \theta)] &= \\ [y_2(t_n; \alpha, \theta) + hL(t_n, y_1(t_n; \alpha, \theta), y_2(t_n; \alpha, \theta))] & \end{aligned} \tag{4.3}$$

The polygon curves:

$$\begin{aligned} y_1(t_n; h; \alpha, \theta) &= \\ [(t_0, y_{1,0}(t_n; \alpha, \theta)), \dots, (t_N, y_{1,N}(t_n; \alpha, \theta))] & \\ y_2(t_n; h; \alpha, \theta) &= \\ [(t_0, y_{2,0}(t_n; \alpha, \theta)), \dots, (t_N, y_{2,N}(t_n; \alpha, \theta))] & \end{aligned} \tag{4.4}$$

The previous equations are Euler approximates  $Y_1(t_n; \alpha, \theta)$  and  $Y_2(t_n; \alpha, \theta)$  when  $t \in [t_0, t_N]$ . Now, to show convergence of these approximates will use the next corollary.

$$\begin{aligned} \lim_{h \rightarrow 0} y_1(t_n; h; \alpha, \theta) &= Y_1(t_n; \alpha, \theta), \\ \lim_{h \rightarrow 0} y_2(t_n; h; \alpha, \theta) &= Y_2(t_n; \alpha, \theta). \end{aligned} \tag{4.5}$$

**Corollary 4.1** If the sequence  $\{H\}_{n=0}^N$  satisfy:

$$|H_{(n+1)}| \leq E |H_n| + Z$$

For some  $E$  and  $Z$  are positive constant, then

$$|H_n| \leq E^n |H_0| + Z \frac{E^n - 1}{E - 1}, \quad n = 0, \dots, N.$$

**Corollary 4.2** If the sequences  $\{H\}_{n=0}^N$  and  $\{G\}_{n=0}^N$ , satisfy:

$$|H_{n+1}| \leq |H_n| + E \max\{|H_n|, |G_n|\} + Z,$$

$$|G_{n+1}| \leq |G_n| + E \max\{|G_n|, |H_n|\} + Z.$$

For some  $E$  and  $Z$  are positive constant and  $U_n = |H_n| + |G_n|, n \in [0, N]$ , then

$$U_n \leq \bar{E}^n U_0 + \bar{Z} \frac{\bar{E}^n - 1}{\bar{E} - 1}$$

Where  $\bar{E} = 1 + 2E$  and  $\bar{Z} = 2Z$ .

Let the functions  $H$  and  $K$  of (3.2), where  $H(t_n, u, v)$  and  $K(t_n, u, v)$  and  $u, v$  are constants such that  $u \leq v$ , then the domain is:

$$M = (t_n, u, v) | t \in [t_0, T], v \in \mathbb{R}, u \in (-\infty, v].$$

**Theorem 4.1** let  $H, Q \in C^1(M)$  and let the partial derivatives of  $H$  and  $Q$  be bounded over  $M$ , then for  $0 \leq \alpha, \theta \leq 1$ , the Euler approximates of (4.4) converge to the solutions  $Y_1(t_n; \alpha, \theta)$  and  $Y_2(t_n; \alpha, \theta)$  uniformly in  $t$ .

**Proof.** Let

$$\lim_{h \rightarrow 0} y_1(t_n; \alpha, \theta) = Y_1(t_n; \alpha, \theta), \quad \lim_{h \rightarrow 0} y_2(t_n; \alpha, \theta) = Y_2(t_n; \alpha, \theta)$$

Where  $t_n = T, n = 0, \dots, N - 1$ . Now, by using Taylor theorem we have:

$$\begin{aligned} [Y_1(t_{n+1}; \alpha, \theta)] &= \left[ \begin{aligned} &Y_1(t_n; \alpha, \theta) + \\ &hH(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta)) \end{aligned} \right] + \\ &\quad \frac{h^2}{2} Y_1''(\zeta_{1,n}) \\ [Y_2(t_{n+1}; \alpha, \theta)] &= \left[ \begin{aligned} &Y_2(t_n; \alpha, \theta) + \\ &hQ(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta)) \end{aligned} \right] + \\ &\quad \frac{h^2}{2} Y_2''(\zeta_{2,n}) \end{aligned} \tag{4.6}$$

Where  $\zeta_{1,n}, \zeta_{2,n} \in (t_n, t_{n+1})$ . Consequently

Let

$$W_n = [Y_1(t_n; \alpha, \theta)] - [y_1(t_n; \alpha, \theta)] \text{ and}$$

$$V_n = [Y_2(t_n; \alpha, \theta)] - [y_2(t_n; \alpha, \theta)]$$

Then

$$W_{n+1} = W_n + h\{H(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta)) - H(t_n, y_1(t_n; \alpha, \theta), y_2(t_n; \alpha, \theta))\} + \frac{h^2}{2} Y_1''(\zeta_{1,n}),$$

$$V_{n+1} = V_n + h\{Q(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta)) - Q(t_n, y_1(t_n; \alpha, \theta), y_2(t_n; \alpha, \theta))\} + \frac{h^2}{2} Y_2''(\zeta_{2,n}).$$

Hence,

$$\begin{aligned} |W_{n+1}| &\leq |W_n| + 2Lh \max\{|W_n|, |V_n|\} + M_1, \\ |V_{n+1}| &\leq |V_n| + 2Lh \max\{|W_n|, |V_n|\} + M_2. \end{aligned} \tag{4.7}$$

Where

$$M_1 = \max_{t_0 \leq t \leq T} Y_1''(t_n; \alpha), \quad M_2 = \max_{t_0 \leq t \leq T} Y_2''(t_n; \alpha)$$

and  $L > 0$  is a bound. Thus, by Corollary 4.2

$$|W_n| \leq (1 + 4Lh)^n |U_0| + h^2 M_1 \frac{(1 + 4Lh)^n - 1}{4Lh},$$

$$|V_n| \leq (1 + 4Lh)^n |U_0| + h^2 M_2 \frac{(1 + 4Lh)^n - 1}{4Lh},$$

Where  $|U_0| = |W_0| + |V_0|$ . In particular

$$|W_N| \leq (1 + 4Lh)^N |U_0| + h^2 M_1 \frac{(1 + 4Lh)^{\frac{T-t_0}{h}} - 1}{4Lh},$$

$$|V_N| \leq (1 + 4Lh)^N |U_0| + h^2 M_2 \frac{(1 + 4Lh)^{\frac{T-t_0}{h}} - 1}{4Lh}.$$

Since  $W_0 = V_0 = 0$ , we get:

$$|W_N| \leq M_1 \frac{e^{4Lh(T-t_0)} - 1}{4L} h,$$

$$|V_N| \leq M_2 \frac{e^{4Lh(T-t_0)} - 1}{4L} h.$$

If  $h \rightarrow 0$  then we have  $W_N \rightarrow 0, V_N \rightarrow 0$ , the proof is done. ■

### 5. Taylor method of order $p$

Let the exact solutions:

$$[Y(t)]_\alpha = [Y_1(t; \alpha, \theta), Y_2(t; \alpha, \theta)]$$

And

$$[y(t)]_\alpha = [y_1(t; \alpha, \theta), y_2(t; \alpha, \theta)]$$

By expansion the Taylor method:

$$x(t + h; \alpha, \theta) = \sum_{i=0}^p \frac{h^i}{i!} x^{(i)}(t; \alpha, \theta), \tag{5.1}$$

Where  $x(t; \alpha, \theta) = Y_1$  or  $Y_2$ . We define



$$F[t, x; \alpha, \theta] = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_1^{(i)}(t, x; \alpha, \theta), \quad (5.2)$$

$$G[t, x; \alpha, \theta] = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_2^{(i)}(t, x; \alpha, \theta), \quad (5.3)$$

The exact and approximate solutions at  $t_n$ , where  $0 \leq n \leq N$ , are denoted by:

$$[Y(t_n)]_\alpha = [Y(t_n; \alpha, \theta)] = [Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta)]$$

And

$$[y(t_n)]_\alpha = [y(t_n; \alpha, \theta)] = [y_1(t_n; \alpha, \theta), y_2(t_n; \alpha, \theta)]$$

Therefore, using the Taylor method and replacing  $Y_1, Y_2$  into (5.1), we have:

$$\begin{aligned} [Y_1(t_{n+1}; \alpha, \theta)] &\approx [Y_1(t_n; \alpha, \theta) + hF(t_n; \alpha, \theta)] \\ [Y_2(t_{n+1}; \alpha, \theta)] &\approx [Y_2(t_n; \alpha, \theta) + hG(t_n; \alpha, \theta)] \end{aligned} \quad (5.4)$$

Where

$$F(t_n; \alpha, \theta) = F(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta))$$

$$G(t_n; \alpha, \theta) = G(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta))$$

We get:

$$\begin{aligned} [y_1(t_{n+1}; \alpha, \theta)] &= \left[ \begin{array}{c} y_1(t_n; \alpha, \theta) + \\ hF(t_n, y_1(t_n; \alpha, \theta), y_2(t_n; \alpha, \theta)) \end{array} \right] \\ [y_2(t_{n+1}; \alpha, \theta)] &= \left[ \begin{array}{c} y_2(t_n; \alpha, \theta) + \\ hG(t_n, y_1(t_n; \alpha, \theta), y_2(t_n; \alpha, \theta)) \end{array} \right] \end{aligned} \quad (5.5)$$

Where

$$y_1(0; \alpha, \theta) = x_1(0; \alpha, \theta), \quad y_2(0; \alpha, \theta) = x_2(0; \alpha, \theta).$$

The polygon curves:

$$\begin{aligned} y_1(t_n; h; \alpha, \theta) &= \left[ (t_0, y_{1,0}(t_n; \alpha, \theta)), \dots, (t_N, y_{1,N}(t_n; \alpha, \theta)) \right] \\ y_2(t_n; h; \alpha, \theta) &= \left[ (t_0, y_{2,0}(t_n; \alpha, \theta)), \dots, (t_N, y_{2,N}(t_n; \alpha, \theta)) \right] \end{aligned} \quad (5.6)$$

Now, to show convergence of these approximates will use the corollary 4.1.

$$\lim_{h \rightarrow 0} y_1(t; \alpha, \theta) = Y_1(t; \alpha, \theta),$$

$$\lim_{h \rightarrow 0} y_2(t; \alpha, \theta) = Y_2(t; \alpha, \theta).$$

Let the functions  $F$  and  $G$  of (5.2) and (5.3), where  $F(t_n, u, v)$  and  $G(t_n, u, v)$  and  $u, v$  are constants such that  $u \leq v$ . That is means:

$$F(t_n, u, v) = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} \min\{f_1^{(i)}(t, \tau) | \tau \in [u, v]\},$$

$$G(t_n, u, v) = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} \max\{f_2^{(i)}(t, \tau) | \tau \in [u, v]\},$$

then the domain is:

$$M = (t_n, u, v) | t \in [t_0, T], v \in \mathbb{R}, u \in (-\infty, v].$$

**Theorem 5.1** let  $H, Q \in C^{p-1}(M)$  and let the partial derivatives of  $H$  and  $Q$  be bounded over  $M$ , then for  $0 \leq \alpha, \theta \leq 1$ , the Taylor approximates of (5.6) converge to the solutions  $Y_1(t_n; \alpha, \theta)$  and  $Y_2(t_n; \alpha, \theta)$  uniformly in  $t$ .

Proof. Let

$$\lim_{h \rightarrow 0} y_1(t_n; \alpha, \theta) = Y_1(t_n; \alpha, \theta)$$

$$\lim_{h \rightarrow 0} y_2(t_n; \alpha, \theta) = Y_2(t_n; \alpha, \theta)$$

Where  $t_n = T, n = 0, \dots, N - 1$ . Now, by using Taylor theorem we have:

$$\begin{aligned} [Y_1(t_{n+1}; \alpha, \theta)] &= \left[ \begin{aligned} &Y_1(t_n; \alpha, \theta) + \\ &hH(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta)) \end{aligned} \right] + \\ &\frac{h^{p+1}}{(p+1)!} Y_1^{(p+1)}(\zeta_{1,n}) \\ [Y_2(t_{n+1}; \alpha, \theta)] &= \left[ \begin{aligned} &Y_2(t_n; \alpha, \theta) + \\ &hQ(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta)) \end{aligned} \right] + \\ &\frac{h^{p+1}}{(p+1)!} Y_2^{(p+1)}(\zeta_{2,n}) \end{aligned} \tag{5.7}$$

Where  $\zeta_{1,n}, \zeta_{2,n} \in (t_n, t_{n+1})$ . Consequently

Let

$$W_n = [Y_1(t_n; \alpha, \theta)] - [y_1(t_n; \alpha, \theta)] \text{ and } V_n = [Y_2(t_n; \alpha, \theta)] - [y_2(t_n; \alpha, \theta)]$$

Then

$$W_{n+1} = W_n + h\{H(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta)) - H(t_n, y_1(t_n; \alpha, \theta), y_2(t_n; \alpha, \theta))\} + \frac{h^{p+1}}{(p+1)!} Y_1^{(p+1)}(\zeta_{1,n}),$$

$$V_{n+1} = V_n + h\{Q(t_n, Y_1(t_n; \alpha, \theta), Y_2(t_n; \alpha, \theta)) - Q(t_n, y_1(t_n; \alpha, \theta), y_2(t_n; \alpha, \theta))\} + \frac{h^{p+1}}{(p+1)!} Y_2^{(p+1)}(\zeta_{2,n}).$$

Hence,

$$\begin{aligned} |W_{n+1}| &\leq |W_n| + 2Lh \max\{|W_n|, |V_n|\} + \frac{h^{p+1}}{(p+1)!} M, \\ |V_{n+1}| &\leq |V_n| + 2Lh \max\{|W_n|, |V_n|\} + \frac{h^{p+1}}{(p+1)!} M. \end{aligned} \tag{4.7}$$

Where

$$M_1 = \max_{t_0 \leq t \leq T} Y_1^{(p+1)}(t_n; \alpha, \theta),$$

$$M_2 = \max_{t_0 \leq t \leq T} Y_2^{(p+1)}(t_n; \alpha, \theta)$$

$M = \max \{M_1, M_2\}$  and  $L > 0$  is a bound. Thus, by Corollary 4.2

$$|W_n| \leq (1 + 4Lh)^n |U_0| + \frac{2h^{p+1}}{(p+1)!} M \frac{(1 + 4Lh)^n - 1}{4Lh},$$

$$|V_n| \leq (1 + 4Lh)^n |U_0| + \frac{2h^{p+1}}{(p+1)!} M \frac{(1 + 4Lh)^n - 1}{4Lh},$$

Where  $|U_0| = |W_0| + |V_0|$ . In particular

$$|W_N| \leq (1 + 4Lh)^N |U_0| + \frac{2h^{p+1}}{(p+1)!} M \frac{(1 + 4Lh)^{\frac{T}{h}} - 1}{2Lh},$$

$$|V_N| \leq (1 + 4Lh)^N |U_0| + \frac{2h^{p+1}}{(p+1)!} M \frac{(1 + 4Lh)^{\frac{T}{h}} - 1}{2Lh}.$$

Since  $W_0 = V_0 = 0$ , we get:

$$|W_N| \leq M \frac{e^{4LT} - 1}{2L(p+1)!} h^p,$$

$$|V_N| \leq M \frac{e^{4LT} - 1}{2L(p+1)!} h^p.$$

Thus, if  $h \rightarrow 0$  we get  $W_N \rightarrow 0, V_N \rightarrow 0$ . ■

## 6. EXAMPLES

**Example 6.1** Consider the complex IVP:

$$\begin{cases} x'(t) = x(t) & t \in [0, 1], \\ x(0) = (0.85 + 0.15\alpha, \theta), (1.25 - 0.125\alpha, \theta) & 0 \leq \alpha, \theta \leq 1. \end{cases}$$

The exact solution at  $t = 1$  is:

$$Y(1; \alpha, \theta) = [(0.85 + 0.15\alpha, \theta)e, (1.25 - 0.125\alpha, \theta)e], \quad 0 \leq \alpha, \theta \leq 1.$$

By using the Taylor method, we have:

$$y_1(t_{n+1}; \alpha, \theta) = y_1(t_n; \alpha, \theta) \sum_{i=0}^p \frac{h^i}{i!}, \quad y_2(t_{n+1}; \alpha, \theta) = y_2(t_n; \alpha, \theta) \sum_{i=0}^p \frac{h^i}{i!}.$$

The exact solution is  $Y_1(t; \alpha, \theta) = y_1(0; \alpha, \theta)e^t, Y_2(t; \alpha, \theta) = y_2(0; \alpha, \theta)e^t$ , at  $t = 1$  is:

$$Y(1; \alpha, \theta) = [(0.85 + 0.15\alpha, \theta) e, (1.25 - 0.25\alpha, \theta) e], \quad 0 \leq \alpha, \theta \leq 1.$$

Where  $\alpha, \theta$  have the same value of  $[0, 1]$ .

**Example 6.2** Consider the complex IVP:

$$\begin{cases} x'(t) = tx(t) \quad [t_0, T] = [-1, 1], \\ x(-1) = (\sqrt{e} - 0.5(1 - \alpha, \theta), \sqrt{e} + 0.5(1 - \alpha, \theta)) \quad 0 \leq \alpha, \theta \leq 1. \end{cases}$$

We have two steps:

Step 1.  $t < 0$ : in this case the parametric form is:

$$x'_1 = tx_2(t; \alpha, \theta), \quad x'_2 = tx_1(t; \alpha, \theta)$$

with the ICs given. The unique solution is:

$$Y_1(t; \alpha, \theta) = \frac{A - B}{2} x_2(0; \alpha, \theta) + \frac{A + B}{2} x_1(0; \alpha, \theta),$$

$$Y_2(t; \alpha, \theta) = \frac{A + B}{2} x_2(0; \alpha, \theta) + \frac{A - B}{2} x_1(0; \alpha, \theta),$$

Where

$$A = e^{\frac{(t^2 - t_0^2)}{2}}, \quad B = \frac{1}{A}.$$

For  $t = 0, t_0 = -1$ , we get:

$$Y_1(0; \alpha, \theta) = \frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{2} x_2(0; \alpha, \theta) + \frac{e^{\frac{1}{2}} + e^{-\frac{1}{2}}}{2} x_1(0; \alpha, \theta),$$

$$Y_2(0; \alpha, \theta) = \frac{e^{\frac{1}{2}} + e^{-\frac{1}{2}}}{2} x_2(0; \alpha, \theta) + \frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{2} x_1(0; \alpha, \theta),$$

and use these complex fuzzy numbers as ICs for step two.

Step 2.  $t \geq 0$ : in this case the parametric form is:

$$x'_1 = tx_1(t; \alpha, \theta), \quad x'_2 = tx_2(t; \alpha, \theta)$$

with the ICs  $Y_1$  and  $Y_2$ , and the solution at  $t > 0$  is:

$$Y_1(t; \alpha, \theta) = Y_1(0; \alpha, \theta)e^{\frac{t^2}{2}}, \quad Y_2(t; \alpha, \theta) = Y_2(0; \alpha, \theta)e^{\frac{t^2}{2}}$$

For  $t = 1$ , we get:

$$Y_1(1; \alpha, \theta) = Y_1(0; \alpha, \theta)e^{\frac{1}{2}}, \quad Y_2(1; \alpha, \theta) = Y_2(0; \alpha, \theta)e^{\frac{1}{2}}$$

Thus,

$$Y_1(1; \alpha, \theta) = \frac{1 - e}{2} x_2(0; \alpha, \theta) + \frac{1 + e}{2} x_1(0; \alpha, \theta),$$

$$Y_2(1; \alpha, \theta) = \frac{e + 1}{2} x_2(0; \alpha, \theta) - \frac{e - 1}{2} x_1(0; \alpha, \theta).$$

We divide  $[-1, 1]$  into  $N$  even equally spaced subintervals to get Euler approximation and define  $y_1(0; \alpha, \theta) = x_1(0; \alpha, \theta)$  and  $y_2(0; \alpha, \theta) = x_2(0; \alpha, \theta)$ :

$$y_1(t_n; \alpha, \theta) = y_1(t_{n+1}; \alpha, \theta) + ht_{n-1}y_2(t_{n-1}; \alpha, \theta), \quad 1 \leq n \leq \frac{N}{2},$$

$$y_2(t_n; \alpha, \theta) = y_2(t_{n+1}; \alpha, \theta) + ht_{n-1}y_1(t_{n-1}; \alpha, \theta), \quad \frac{N}{2} + 1 \leq n \leq N.$$

## CONCLUSIONS

This work we concerned with methods of solving the complex fuzzy initial value problem. We first developed our new solution  $Y(t)$ . Basically,  $Y(t)$  is the complex fuzzification of the crisp solution to the initial value problem. We gave necessary and sufficient conditions for  $Y(t)$  to solve the complex fuzzy initial value problem. And we consider Taylor and Euler for complex fuzzy differential equations, and we establish some basic definitions and theorems to build our result. In this work we obtained some results that are necessary to develop fuzzy differential equations and to be well-to-do.

## REFERENCES

- [1] Abdallah Al-Husban and Abdul Razak Salleh, "Complex Fuzzy Group Based on Complex Fuzzy Spac", Global Journal of Pure and Applied Mathematics. ISSN 0973-1768 Volume 12, Number 2 pp. 1433-1450, 2016,
- [2] Alkouri, A., & Salleh, A. "Complex intuitionistic fuzzy sets". American Institute of Physics. *In AIP Conference Proceedings* (Vol. 1482, No. 1, pp. 464-470. 2012. <https://doi.org/10.1063/1.4757515>
- [3] Alkouri, A., & Salleh, A. "Complex Atanassov's intuitionistic fuzzy relation". *In Abstract and Applied Analysis* (Vol. 2013). (2013a) Hindawi <https://doi.org/10.1155/2013/287382>
- [4] Alkouri, A., & Salleh, A. "Linguistic variable, hedges and several distances on complex fuzzy sets". *Journal of Intelligent & Fuzzy Systems*, 26(5), 2527-2535. (2014). DOI: 10.3233/IFS-130923
- [5] D. Karpenko, R.A. Van Gorder, A. Kandel, "The Cauchy problem for complex fuzzy differential equations", *Fuzzy Sets Syst.* 245 18–29. 2014 <https://doi.org/10.1016/j.fss.2013.11.001>
- [6] D.E. Tamir, L. Jin, A. Kandel, "A new interpretation of complex membership grade", *Int. J. Intell. Syst.* 26 285–312. 2011. <https://doi.org/10.1002/int.20454>
- [7] Jun, M., Zhang, G. & Lu, J. "A method for multiple periodic factor prediction problems using complex fuzzy sets." *IEEE Transaction on Fuzzy Systems* 20(1): 32-45, 2012. <https://doi.org/10.1109/TFUZZ.2011.2164084>
- [8] L. Zadeh, "Fuzzy sets", *Inform. and Control* 8 338-353., 1965. [https://doi.org/10.1016/S0019-9958\(65\)90241-X](https://doi.org/10.1016/S0019-9958(65)90241-X)
- [9] O. Kaleva, "Fuzzy differential equations", *Fuzzy Sets Syst.* 24 301–307, 1987. [https://doi.org/10.1016/0165-0114\(87\)90029-7](https://doi.org/10.1016/0165-0114(87)90029-7)
- [10] Ramot, D., Friedman, M., Langholz, G. & Kandel, A. "Complex fuzzy logic". *IEEE Transaction on Fuzzy Systems* 11(4): 450-461, 2003. <https://doi.org/10.1109/TFUZZ.2003.814832>
- [11] Ramot, D., Milo, R., Friedman, M., & A. Kandel, A.. "Complex fuzzy sets" *.IEEE Transaction on Fuzzy Systems*10: 171-186, 2002. <https://doi.org/10.1109/91.995119>
- [12] S. Abbasbandy, T.A. Viranloo, "Numerical solutions of fuzzy differential equations by taylor method, *Comput. Methods Appl. Math.* 2 (2) 113– 124, 2002. <https://doi.org/10.2478/cmam-2002-0006>
- [13] Tamir, D.E. & Kandel, A. "Axiomatic theory of complex fuzzy logic and complex fuzzy classes." *Int. J. of Computers, Communications & Control* 6(3): 562-576, 2011.
- [14] Tamir, D.E., Last, M. & Kandel, A.. Generalized complex fuzzy propositional logic. Accepted for publication in the World Conference on Soft computing, San Francisco. (2012a)
- [15] Tamir, D.E., Last, M., Kandel, A. The Theory and Applications of Generalized Complex Fuzzy Propositional Logic. In: Yager, R., Abbasov, A., Reformat, M., Shahbazova, S. (eds) *Soft Computing: State of the Art Theory and Novel Applications*. Studies in Fuzziness and Soft Computing, vol 291. Springer, Berlin, Heidelberg. [https://doi.org/10.1007/978-3-642-34922-5\\_13](https://doi.org/10.1007/978-3-642-34922-5_13) (2013)
- [16] D. E. Tamir, H. N. Teodorescu, M. Last and A. Kandel, "Discrete complex fuzzy logic," 2012 Annual Meeting of the North American

Fuzzy Information Processing Society (NAFIPS), Berkeley, CA, USA, , pp. 1-6, (2012b) doi: 10.1109/NAFIPS.2012.6291020.

- [17] Zhang G, Dillon ST, Cai YK, Ma J, Lu J “Operation properties and d-equalities of complex fuzzy sets.” Int J Approx Reason 50:1227–1249, 2009. <https://doi.org/10.1016/j.ijar.2009.05.010>
- [18] Zhifei, C., Sara, A., James, M. & Scott, D. ANCFIS: a neurofuzzy architecture employing complex fuzzy set. IEEE Transactions on Fuzzy Systems 19: 305-322, 2012. <https://doi.org/10.1109/TFUZZ.2010.2096469>

DOI: <https://doi.org/10.15379/ijmst.v10i3.1539>

This is an open access article licensed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>), which permits unrestricted, non-commercial use, distribution and reproduction in any medium, provided the work is properly cited.