On Some Properties of Cylindrically Transformed Systems With *R(π)* Symmetry and Phase Dynamics

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Abstract: Nonlinear dynamical systems with R(π) symmetry are shown to behave in a very interesting manner under a new transformation of dynamical variables. Such property helps to identify the phase dynamics embedded in the system but preserves the basic property of the attractor intact. This is very similar to those phenomenon discussed with the help of covering transformation in the literature. The Poincare sections obtained are identical to those obtained through covering transformation and hence indicate to a similar topological structure and identical dynamical characteristics.

Keywords: Cylindrical transformation, Phase dynamics, Cover, R(π) symmetry.

1. INTRODUCTION

In recent times it has been found that a wide class of physical phenomena in physics, biology and medicine can be either modeled or explained with the help of a globally coupled network of phase oscillator. Such models were initially discussed by Kuramoto [1], WinFree [2],and Watanabe [3] and some other authors. These special type of phase oscillators with nonlinear couplings between them- selves do play a very important role for studying the process of synchronization in a very large net- work of such objects which has very important implications in explaining the phenomena of pedestrians on a foot bridge[4], neuronal population behavior [5] etc. On the other hand in the usual study of nonlinear dynamical system the models we use have both the amplitude and phase dynamics entangled together. So one possible approach is to derive the equation of the corresponding phase dynamics by a standard procedure say variational approach. Our goal in this paper is to suggest an alternative approach to study such phase in standard nonlinear 3D models. In this respect it may be noted that the detection of phase either from the nonlinear equation itself or from the corresponding data both are not very straight forward. Identification of the phase of the chaotic time series was a perineal problem for a long time. Initial break through regarding its identification and analysis of phase synchronization was done by Pikovsky, Rosenblum, Kurth [6]. The identification of phase have been the easiest in case of Rosseler attractor which has a a single center of rotation. This is clearly seen from the projection in phase space. On the other hand if no such visual or geometrical aid is available, the famous approach of empirical mode de-composition, discovered by Huang *et.al*. [7], is seen to be applicable. But it is a well-known fact that the Lorenz attractor shows two center of rotation. Thus technique applicable to Rosseler system does not hold well in Lorenz system [8]. But a simple trick can do the job. One just gets the attractor in the $\left(u=\sqrt{x^2+y^2}\right)$ plane, where it exhibits one center of rotation and Rosseler approach becomes applicable, in these new variables. So it was our idea to study the Lorenz system in these new set of variables;

$$
\begin{aligned}
u &= \sqrt{x^2+y^2}\\
v &= \tan^{-1}\frac{y}{x}\\
z &= z
\end{aligned}
\tag{1}
$$

Where the mother equation is written as

$$
\begin{aligned}
\dot{x} &= \sigma(y-x)\\
\dot{y} &= r\,x - y - x\,z\\
\dot{z} &= x\,y - b\,z
\end{aligned}
\tag{2}
$$

It is observed that the new nonlinear system, in the variables (u, v, z), shows some remarkable prop- erties along with the fact that it has an well defined phase from the beginning. In general the transformation of non-linear systems were done long back by Miranda and Stone [9] to determine some cover or dou- ble cover of dynamical system to see the mapping between different forms of attractors in different phase space variables. It may be mentioned that after a lapse of almost seven years the same transformations were used by Gilmore and Letellier [10, 11] to study in detail the topological properties associated with the chaotic systems. They have made some important observations regarding the structure of templates,

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under such covering transformation. The most important and significant fact is that though our transformation is different from the type discussed in reference [10, 11], yet it has got some interesting consequences for the new non-linear systems so produced.

Incidentally we have also applied this transformation to Bark-Shaw [12] and Dual Lorenz equation, and observed that they behave in the same manner. In this connection it may be mentioned that in the paper of C. Letellier *et al*. [10, 11], it was observed that simple symmetry properties of non-linear equations can help to classify the different nature of the attractors and give some clues to their properties. The Lorenz equation is known to have symmetry; $(x,y,z) \to (-x,-y,z)$ and some other equations also do belong to this class. C Letellier *et al*.[10] called such a system to have $R(\pi)$ symmetry. It is easy to observe that Bark Shaw and Dual Lorenz also behave in same fashion.

In this communication we have transformed each of these equations with the help of transformations defined by Eq. (1) and have analyzed them completely. It is observed that other than the determination of the phase directly which was the main contention, the Poincare map of the transformed Lorenz and other equations under consideration behaves exactly in the same way as observed by G Bryne *et al*.[11], though the equations obtained after the transformations are widely different.

2. FORMULATION

Consider the Lorenz system written in Eq. (2) and let us make the change of variable given by Eq. (1), whence one gets

$$\begin{aligned}
\dot{u} &= \frac{\sigma+r-z}{2} u\sin(2v) - \sigma u\cos^2(v) - u\sin^2(v) \\
\dot{v} &= \frac{\sigma-1}{2}\sin(2v) + (r-z)\cos^2(v) - \sigma\sin^2 v \\
\dot{z} &= \frac{u^2}{2}\sin(2v) - bz
\end{aligned} \tag{3}$$

The fixed point equation leads to $(u=0, v=\frac{\pi}{2}, z=0)$ and $(u=0, v=-\tan^1\frac{1}{\sigma}, z=0)$. The corresponding Jacobian around the fixed point is

$$j = \begin{pmatrix} \frac{\sigma+r-z}{2}\sin(v) - \sigma\cos^z(v) - u\sin^z(v) & A & -u\frac{\sin(2v)}{2} \\ 0 & B & \cos^z(v) \\ u\sin(2v) & u^z\cos(2v) & -b \end{pmatrix} \tag{4}$$

where

$$\begin{aligned}
A &= (\sigma+r-z)u\cos(2v) + \sigma u\sin(2v) - u\sin(2v) \\
B &= (\sigma-1)\cos(2v) - (r-z)\sin(2v) - \sigma\sin(2v)
\end{aligned} \tag{5}$$

For the critical condition $(0,\frac{\pi}{4},0)$, the eigenvalues are $\left(-b, -(r+\sigma), \frac{(r-1)}{2}\right)$. For the second case the eigenvalues are

$$\begin{aligned}
\lambda_1 &= \frac{\sigma^3-\sigma^2-r\sigma-1}{1+\sigma^2} \\
\lambda_2 &= \frac{\left(\sigma^3+\sigma^2+(2r-1)\sigma+1\right)}{1+\sigma^2} \\
\lambda_3 &= -b
\end{aligned} \tag{6}$$

The nature of chaotic behavior is very simply reflected in these values when numerical solutions are done for (σ, r).

Our numerical simulation starts with the phase space diagram of the attractor shown in (Figure **1b**). For the parameter values $r=28.0, \sigma=10.0, b=\frac{8}{3}$. It is interesting to observe that the form of the attractor in its projection in the (u,v) plane shows a single center of rotation insisted of two center in the original Lorenz equation (Figure **1a**). It may be pointed out that the same kind of result was reproduced by Gilmore *et al*. and Miranda *et al.* for Lorenz system. To illustrate the monotonic behavior of the variable v, its time series is displayed in Figure (**2**). In Figure (**3**), we show the bifurcation diagram for u with respect to the parameter σ, which shows clearly the positions of periodic windows.

To characterize the nature of the new system Eq. (3), we next compute all the Lyapunov coefficient for the parameter values $r=28.0, \sigma=10.0, b=\frac{8}{3}$, these are depicted in Figure (**4**). One can easily observe that we have one positive, another almost zero and third one negative which was basic property of the Lorenz system. To extract more interesting properties of the transformed system, we have computed the Poincare section for $r=28.0\ and\ r=65.584$. Other parameter values are kept the above mentioned values. The

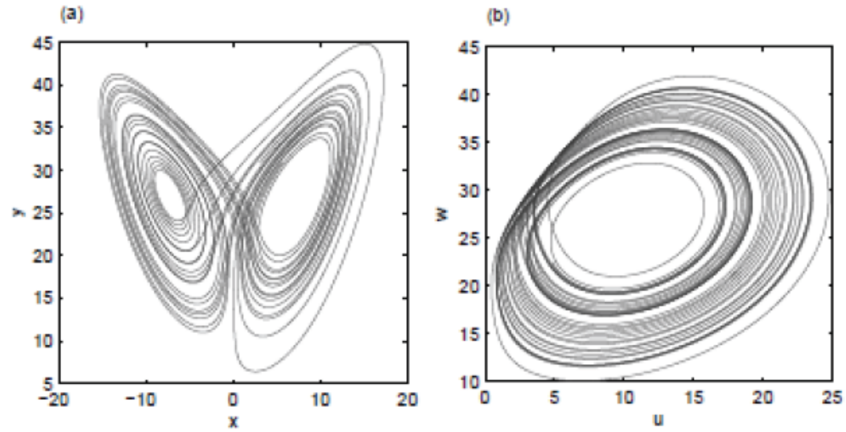

Figure 1: (**a**) Lorenz attractor at $r = 28.0, \sigma = 10.0, b = \frac{8}{3}$. (**b**) Attractor for the transformed Lorenz system, given by equation-3, at $\sigma = 10.0, r = 28.0, b = \frac{8}{3}$.

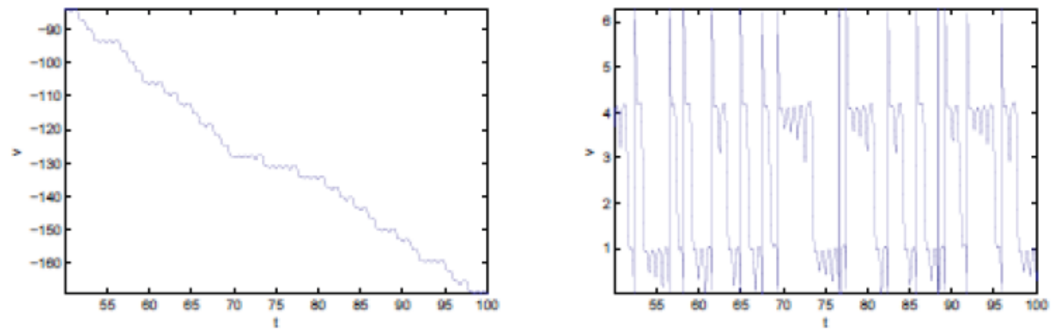

Figure 2: (**a**) Variation of v with time. (**b**) Variation of $Mod(v, 2\pi)$ *with time*

corresponding situations are shown in Figures (**5a**) and (**5b**). One should note the striking similarity of this figure with that obtained in [11] for same values of r which are obtained through covering transformation. As is well known that similarity of Poincare map gives same set of symbolic sequence and in turn will lead to similar types of template.

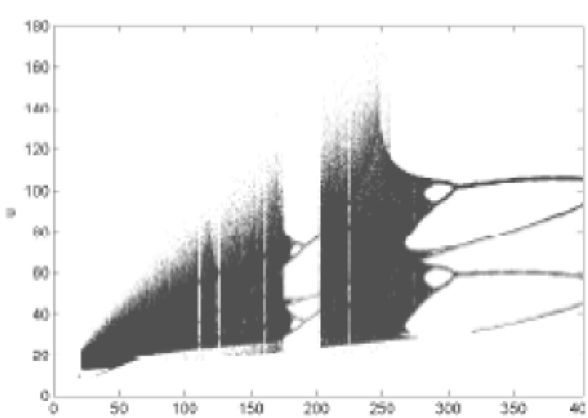

Figure 3: The bifurcation diagram of transformed system for $\sigma \in [0.0, 400.0], r = 28.0, b\frac{8}{3}$.

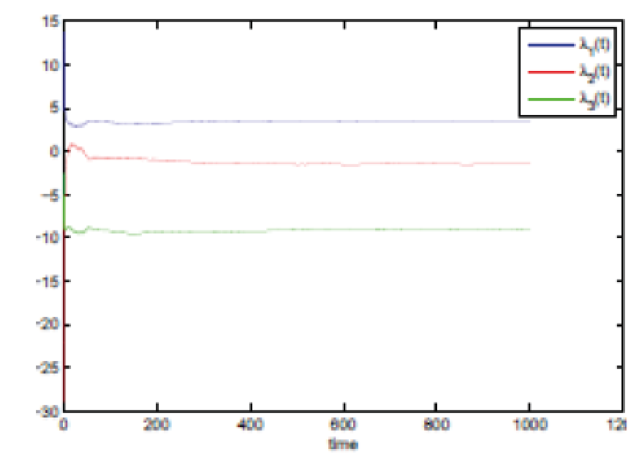

Figure 4: Lyapunov exponent of the transformed system for σ=10.0,r=28.0,b=8/3

Next consider Burke-Shaw system written as,

$$\begin{aligned} \dot{x} &= -s(x + y) \\ \dot{y} &= -y - s\,x \\ \dot{z} &= s\,x\,y + v \end{aligned} \tag{7}$$

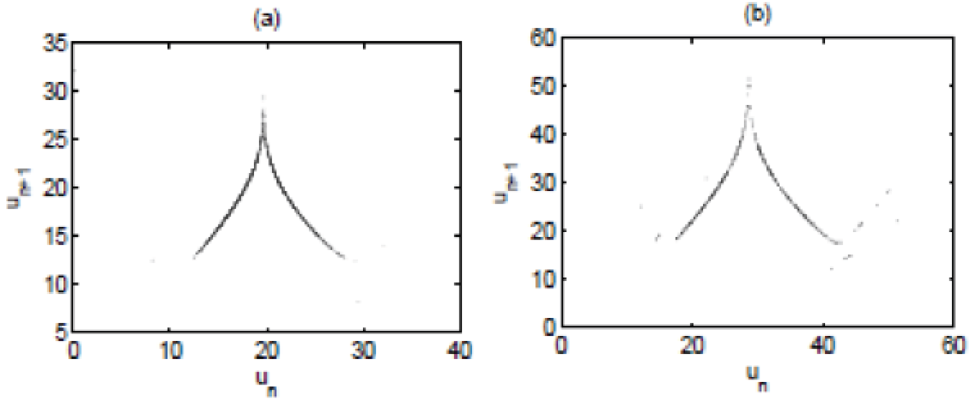

Figure 5: Poincare section for r=28.0 and r=65.584 of Eq.(3).

One can immediately see that Eq.(5) has the same symmetry $(x,y,z)\rightarrow(-x,-y,z)$. The transformed equation in this case is written as

$$\begin{aligned}
\dot{u} &= -\frac{s(1+z)}{2}u\sin(2v) - s\,u\cos^2(v) - u\sin^2(v)\\
\dot{v} &= \frac{s-1}{2}\sin(2v) + s\sin^2(v) - s\,z\cos^2(v)\\
\dot{z} &= \frac{s}{2}u^2\sin(2v) + V
\end{aligned} \tag{8}$$

In Figure (**6**), we show the (u,z)projection of its attractor for the parameter values $(s=10.0, v=4.271)$. This again shows a single center of rotation. It may be added further that same phenomena is repeated in case of Lᴜ system [13].

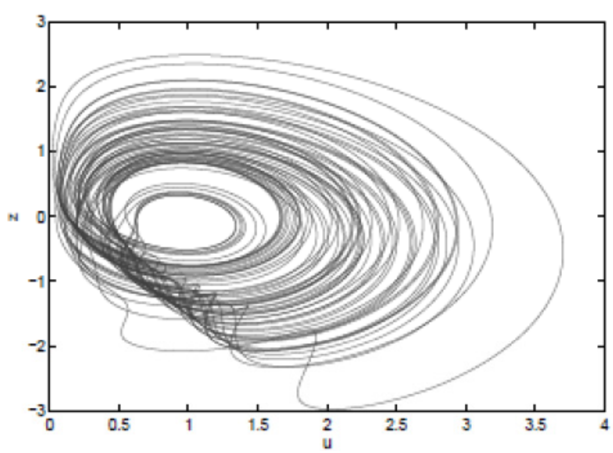

Figure 6: Attractor of the transformed system of Bark-Shaw attractor at s=10.0 and v=4.2

3 CONCLUSION AND DISCUSSION

In our above discussions we have studied a very simple cylindrical transformation for some 3D systems with R(π) symmetry. Our transformation has the unique feature that it unfolds the phase dynamics embedded inside the system. Topologically speaking it exhibits features similar to the covering transformation already discussed in the literature. These properties are analyzed with the study of the corresponding Poincare map and Lyapunov exponent. It may be added that existence of similar type of Poincare map implies a similar type of symbolic dynamics which in turn controls the dynamics of the attractor. An interesting and important application of this transformation will be in the analysis of phase synchronization in coupled nonlinear oscillator systems. Such study is in progress and will be reported in future communication. In each case, the transformed system exhibits the same feature as obtained via covering transformation discussed in detail previously. The similarity is exhibited through the detailed study of the Poincare map and the Lyapunov exponent.

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